DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -III

PARTIAL DIFFERENTIAL EQUATIONS

DEMATH3OLEC3

BLOCK-2

UNIVERSITY OF NORTH BENGAL

Postal Address:

The Registrar,

University of North Bengal,

Raja Rammohunpur,

P.O.-N.B.U., Dist-Darjeeling,

West Bengal, Pin-734013,

India.

Phone: (O) +91 0353-2776331/2699008

Fax: (0353) 2776313, 2699001

Email: regnbu@sancharnet.in ; regnbu@nbu.ac.in

Wesbsite: www.nbu.ac.in

First Published in 2019

FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

PARTIAL DIFFERENTIAL EQUATIONS

BLOCK-1

BLOCK-2

BLOCK-2 PARTIAL DIFFERENTIAL EQUATIONS

Introduction to Block

The world is based to a large content on Partial Differential Equations. In this book explained how Partial Differential Equations are useful in day to day life.

Examples are the vibrations of solids, the flow of fluids, the diffusion of chemicals, the speed of the heat, the structure of molecules and radiation of electromagnetic waves. Provided most important proofs and solved examples.

This block contents Elliptic Partial Differential Equations, The heat and Schrodinger equations, Solutions of wave equations and Separation of variables.

UNIT-8 ELLIPTIC PDES

STRUTURE

8.0 OBJECTIVE

In this we will learn and understand about Weak formulation of Dirichlet problem, Variational formulation, The space, The Poincare inequality, Existence of weak solution of the Dirichlet problem, General linear, second order elliptic.

8.1 INTRODUCTION

One of the main advantages of extending the class of solutions of a PDE from classical solutions with continuous derivatives to weak solutions with weak derivatives is that it is easier to prove the existence of weak solutions. Having established the existence of weak solutions, one may then study their properties, such as uniqueness and regularity, and perhaps prove under appropriate assumptions that the weak solutions are, in fact, classical solutions. There is often considerable freedom in how one defines a weak solution of a PDE; for example, the function space to which a solution is required to belong is not given a priori by the PDE itself. Typically, we look for a weak formulation that reduces to the classical formulation under appropriate smoothness assumptions and which is amenable to a mathematical analysis; the notion of solution and the spaces to which solutions belong are dictated by the available estimates and analysis.

8.2 WEAK FORMULATION OF THE DIRICHLET PROBLEM

Let us consider the Dirichlet problem for the Laplacian with homogeneous boundary conditions on a bounded domain Ω in R^n,

(8.1) $$-\Delta u = f \text{ in } \Omega,$$

(8.2) $$u = 0 \text{ on } \partial\Omega$$

First, suppose that the boundary of Ω is smooth and u, $f : \overline{\Omega} \to R$ are smooth functions. Multiplying (8.1) by a test function φ, integrating the result over Ω, and using the divergence theorem, we get

(8.3) $$\int_\Omega Du.\phi dx = \int_\Omega f\, \phi dx \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

The boundary terms vanish because $\phi = 0$ on the boundary. Conversely, if f and Ω are smooth, then any smooth function u that satisfies (8.3) is a solution of (8.1). Next, we formulate weaker assumptions under which (8.3) makes sense. We use the flexibility of choice to define weak solutions with L^2-derivatives that belong to a Hilbert space; this is helpful because Hilbert spaces are easier to work with than Banach spaces. Furthermore, it leads to a variational form of the equation that is symmetric in the solution *u* and the test function ϕ. Our goal of obtaining a symmetric weak formulation also explains why we only integrate by parts once in (8.3). We briefly discuss some other ways to define weak solutions at the end of this section.

We would need to use Banach spaces to study the solutions of Laplace's equation whose derivatives lie in L^p for $p \neq 2$, and we may be forced to use Banach spaces for some PDEs, especially if they are nonlinear.

By the Cauchy-Schwartz inequality, the integral on the left-hand side of (8.3) is finite if Du belongs to $L^2(\Omega)$, so we suppose that $u \in H^1(\Omega)$. We impose the boundary condition (8.2) in a weak sense by requiring that $u \in H_0^1(\Omega)$. The left hand side of (8.3) then extends by continuity to $\phi \in H_0^1(\Omega) = C_c^\infty(\Omega)$. The right hand side of (8.3) is well-defined for $\phi \in H_0^1(\Omega)$ if $f \in L^2(\Omega)$, but this is not the most general f for which it makes sense; we can define the right-hand for any *f* in the dual space of $H_0^1(\Omega)$.

DEFINATION 8.1.

The space of bounded linear maps $f : H_0^1(\Omega) \to R$ is denoted by $H^1(\Omega) = H_0^1(\Omega)^*$, and the action of $f \in H^{-1}(\Omega)$ on $\phi \in H_0^1(\Omega)$ by $\langle f, \phi \rangle$. The norm of $f \in H^{-1}(\Omega)$ is given by

$$\| f \|_{H^{-1}} = \sup \left\{ \frac{|\langle f, \phi \rangle|}{\| \phi \|_{H_0^1}} : \phi \in H_0^1, \phi \neq 0 \right\}$$

A function $f \in L^2(\Omega)$ defines a linear functional $F_f \in H^{-1}(\Omega)$ by

$\langle F_f, \upsilon \rangle = \int f\upsilon dx = (f, \upsilon)_{L^2}$ for all $\upsilon \in H_0^1(\Omega)$

Here $(.,.)$ L_2 denotes the standard inner product on $L^2(\Omega)$. The functional F_f is bounded on $H_0^1(\Omega)$ with $\| F_f \|_{H^{-1}} \leq \| f \|_{L^2}$ since, by the Cauchy-Schwartz inequality,

$$|\langle F_f, \upsilon \rangle| \leq \| f \|_{L^2} \| \upsilon \|_{L^2} \leq \| f \|_{L^2} \| \upsilon \|_{H_0^1}.$$

We identify F_f with *f*, and write both simply as *f*.

Such linear functionals are, however, not the only elements of $H^{-1}(\Omega)$. As we will show below, $H^{-1}(\Omega)$ may be identified with the space of distributions on Ω that are sums of first-order distributional derivatives of functions in $L^2(\Omega)$. Thus, after identifying functions with regular distributions, we have the following triple of Hilbert spaces

$$H_0^1(\Omega) \to L^2(\Omega) \to H^{-1}(\Omega), \quad H^{-1}(\Omega) = H_0^1(\Omega)$$

Moreover, if $f \in L^2(\Omega) \subset H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$,

then $\langle f, u \rangle = (f, u)_{L^2}$,

So the duality pairing coincides with the L^2-inner product when both are defined. This discussion motivates the following definition.

DEFINITION 8.2

Let Ω be an open set in R^n and $f \in H^{-1}(\Omega)$. A function $u : \Omega \to R$ is a weak solution of (8.1) – (8.2) if : $(a) u \in H_0^1(\Omega); (b)$

(8.4) $$\int_\Omega Du.D\phi dx = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

Here, strictly speaking, 'function' means an equivalence class of functions with respect to pointwise a.e. equality.

We have assumed homogeneous boundary conditions to simplify the discussion. If Ω is smooth and $g : \partial\Omega \to R$ is a function on the boundary that is in the range of the trace map

$T : H^1(\Omega) \to L^2(\partial\Omega)$, say $g = Tw$, then we obtain a weak formulation of the non-homogeneous Dirichet problem

$-\Delta u = f$ in Ω,

$u = g$ on $\partial\Omega$,

by replacing (a) in Definition 8.2 with the condition that $u - w \in H_0^1(\Omega)$ The definition is otherwise the same. The range of the trace map on $H^1(\Omega)$ for a smooth domain Ω is the fractional-order Sobolev space $H^{1/2}(\partial\Omega)$; thus if the boundary data g is so rough that $g \notin H^{1/2}(\partial\Omega)$, then there is no solution $u \in H^1(\Omega)$ of the nonhomogeneous BVP.

Finally, we comment on some other ways to define weak solutions of Poisson's equation. If we integrate by parts again in (8.3), we find that every smooth solution u of (8.1) satisfies

(8.5) $$-\int_\Omega u\Delta\phi dx = \int_\Omega f\phi dx \text{ for all } \phi \in C_c^\infty(\Omega)$$

This condition makes sense without any differentiability assumptions on *u*, and we can define a locally integrable function $u \in L^1_{loc}(\Omega)$) to be a weak solution of $-\Delta u = f$ for $f \in L^1_{loc}(\Omega)$ if it satisfies (8.5). One problem with using this definition is that general functions $u \in L^p(\Omega)$ do ot have enough regularity to make sense of their boundary values on $\partial\Omega^2$.

More generally, we can define distributional solutions $T \in D^{'}(\Omega)$ of Poisson'sequation $-\Delta T = f$ with $f \in D'(\Omega)$ by

(8.6) $$-\langle T, \Delta\phi\rangle = \langle f, \phi\rangle \text{ for all } \phi \in C_c^\infty(\Omega)$$

While these definitions appear more general, because of elliptic regularity they turn out not to extend the class of variational solutions we consider here if $f \in H^{-1}(\Omega)$, and we will not use them below.

8.3 VARIATIONAL FORMULATION

DEFINITION 8.2

Of a weak solution in is closely connected with the variational formulation of the Dirichlet problem for Poisson's equation. To explain this connection, we first summarize some definitions of the differentiability of functionals (scalar-valued functions) acting on a branch space.

DEFINITION 8.3.

A functional $J : X \to R$ on a branch space X is differentiable at $x \in X$ if there is a bounded linear functional $A : X \to R$ such that

$$\lim_{h\to 0} \frac{|K(x+h) - J(x) - Ah|}{\|h\| x} = 0$$

If A exists, then it is unique, and it is called the derivative, or differentiable, of J at x, denoted $DJ(x) = A$.

This definition expresses the basic idea of a differentiable function as one which can be approximated locally by a linear map. If J is differentiable at every point of X, then $DJ : X \to X^*$ maps $x \in X$ to the linear functional $DJ(x) \in X^*$ that approximates J near x^2 For example, if Ω is bounded and $\partial\Omega$ is smooth, then point wise evaluation $\phi \to \phi|_{\partial\Omega}$ on $C(\overline{\Omega})$ extends to a bounded, linear trace map $T : H^s(\Omega) \to H^{s-1/2}(\Omega)$ if $s > 1/2$ but not if $s \le 1/2$. In particular, there is no sensible definition of the boundary values of a general function

$u \in L^2(\Omega)$. We remark, however, that if $u \in L^2(\Omega)$ is a weak solution of $-\Delta u = f$ where

$f \in L^2(\Omega)$, then elliptic regularity implies that $u \in H^2(\Omega)$, so it does have a well-defined boundary value $u|_{\partial\Omega} \in H^{3/2}(\partial\Omega)$; on the other hand, if $f \in H^{-2}(\Omega)$, then $u \in L^2(\Omega)$ and we cannot make sense of $u|_{\partial\Omega}$.

A weaker notion of differentiability (even for functions $J : R^2 \to R$ – see

Example 8.8 is the existence of directional derivatives

$$\delta J(x;h) = \lim_{\in \to 0}\left[\frac{J(x+\in h) - J(x)}{\in}\right] = \frac{d}{d\in} J(x+\in h)|_{\in=0}$$

If the directional derivative at x exists for every $h \in X$ and is a bounded linear functional on h, then $\delta J(x;h) = \delta J(x)h$ where $\delta J(x) \in X^*$. We call $\delta J(x)$ the Gateaux derivative of J at x. The derivative DJ is then called the Frechet derivative to distinguish it from the directional or Gateaux derivative. If J is differentiable at x, then it is Gateaux-differentiable at x and $DJ(x) = \delta J(x)$, but the converse is not true.

Example 8.4

Define $f : R^2 \to R$ by $f(0,0) = 0$ and $f(x,y) = \left(\frac{x^2}{x^2 + y^4}\right)^2$ if $(x,y) \ne (0,0)$.

Then f is Gateaux-differentiable at 0, with $\delta f(0) = 0$, but f is not frechet differentiable at 0.

Notes

If $J: X \to R$ attains a local minimum at $x \in X$ and J is differentiable at x then for every $h \in X$ the function $J_{x:h}: R \to R$ defined by $J_{x;h}(t) = J(x+th)$ is differentiable at $t = 0$ and attains a minimum at $t = 0$. It follows that

$$\frac{d.J_{x;h}}{dt}(0) = \delta J(x;h) = 0 \text{ for every } h \in X.$$

Hence $DJ(x) = 0$. Thus, just as in multivariable calculus, an extreme point of a differentiable functional is a critical point where the derivative is zero. Given $f \in H^{-1}(\Omega)$, define a quadratic functional $J: H_0^1(\Omega) \to R$ by

(8.7) $$J(u) = \frac{1}{2}\int_\Omega |Du|^2 \, dx - \langle f, u \rangle$$

Clearly, J is well-defined.

Proposition 8.5.

The functional $J: H_0^1(\Omega) \to R$ in (8.7) is differentiable. Its derivative $DJ(u): H_0^1(\Omega) \to R$ at $u \in H_0^1(\Omega)$ is given by

$$DJ(u)h = \int_\Omega Du.Dh \, dx - \langle f, h \rangle \quad \text{for } h \in H_0^1\Omega$$

Proof. Given $u \in H_0^1(\Omega)$, define the linear map $A: H_0^1(\Omega) \to R$ by

$$Ah = \int_\Omega Du.Dh \, dx - \langle f, h \rangle$$

Then A is bounded, with $\| A \| \le \| Du \|_{L^2} + \| f \|_{H^{-1}}$, since

$$|Ah| \le \| Du \|_{L^2} \| Dh \|_{L^2} + \| f \|_{H^{-1}} \| h \|_{H_0^1} \le \left(\| Du \|_{L^2} + \| f \|_{H^{-1}} \right) \| h \|_{H_0^1} .$$

For $h \in H_0^1(\Omega)$, we have

$$J(u+h) - J(u) - Ah = \frac{1}{2}\int_\Omega |Dh|^2 \, dx.$$

It follows that $| J(u+h) - J(u) - Ah | \le \frac{1}{2} \| h \|_{H_0^1}^2$,

8.4 THE SPACE $H^{-1}(\Omega)$

And therefore

$$\lim_{h\to 0}\frac{\left|J(u+h)-J(u)-Ah\right|}{\|h\|_{H_0^1}}=0$$

Which proves that J is differentiable on $H_0^1(\Omega)$ with $DJ(u)=A$.

Note that $DJ(u) = 0$ if and only if u is a weak solution of Poisson's equation in the sense of Definition 8.2. Thus, we have the following result

Corollary 8.6. $J:H_0^1(\Omega)\to R$ defined in (8.7) attains a minimum at $u\in H_0^1(\Omega)$, then u is a weak solution of $-\Delta u=f$ in the sense of **Definition 8.2.**

In the direct method of the calculus of variations, we prove the existence of a minimizer of *J* by showing that a minimizing sequence $\{u_n\}$ converges in a suitable sense to a minimizer u. This minimizer is then a weak solution of (8.1)–(8.2). We will not follow this method here, and instead establish the existence of a weak solution by use of the Riesz representation theorem. The Riesz representation theorem is, however, typically proved by a similar argument to the one used in the direct method of the calculus of variations, so in essence the proofs are equivalent.

The negative order Sobolev space $H^{-1}(\Omega)$ can be described as a space of distributions on **Ω**.

THEOREM 8.7.

The space $H^{-1}(\Omega)$ consists of all distributions $f\in D'(\Omega)$ of the form

(8.8) $$f=f_0+\sum_{i=1}^{n}\partial_i f_i \quad \text{where } f_0, f_i\in L^2(\Omega)$$

These distributions extend uniquely by continuity from D(Ω) to bounded linear functionals on $H_0^1(\Omega)$. Moreover,

(8.9) $$\|f\|_{H^{-1}(\Omega)}=\inf\left\{\left(\sum_{i=0}^{n}\int_{\Omega} f_i^2 dx\right)^{1/2} :\text{such that } f_0, f_i \text{ satisfy } (4.8)\right\}$$

Notes

PROOF. First suppose that $f \in H^{-1}(\Omega)$. By the Riesz representation theorem there is a function $g \in H_0^1(\Omega)$ such that

(8.10) $$\langle f, \phi \rangle = (g, \phi)_{H_0^1} \text{ for all } \phi \in H_0^1(\Omega)$$

Here, $(.,.)_{H_0^1}$ denotes the standard inner product on $H_0^1(\Omega)$

$$(u, \upsilon)_{H_0^1} = \int_\Omega (u\upsilon + Du.D\upsilon)\,dx.$$

Identifying a function $g \in L^2(\Omega)$ with its corresponding regular distribution, restricting f to $\phi \in D(\Omega) \subset H_0^1(\Omega)$, and using the definition of the distributional derivative, we have

$$\langle f, \phi \rangle = \int_\Omega g\phi\,dx + \sum_{i=1}^{n} \int_\Omega \partial_i g \partial_i \phi\,dx$$

$$= \langle g, \phi \rangle + \sum_{i=1}^{n} \langle \partial_i g, \partial_i \phi \rangle$$

$$= \left\langle g - \sum_{i=1}^{n} \partial_i g_i, \phi \right\rangle \qquad \text{for all } \phi \in D(\Omega)$$

where $g_i = \partial_i g \in L^2(\Omega)$. Thus the restriction of every $f \in H^{-1}(\Omega)$ from $H_0^1(\Omega)$ to $D(\Omega)$ is a distribution

$$f = g - \sum_{i=1}^{n} \partial_i g_i$$

of the form (8.8). Also note that taking $\varphi = g$ in (8.10), we get $\langle f, g \rangle = \| g \|_{H_0^1}^2$ which implies that

$$\| f \|_{H^{-1}} \geq \| g \|_{H_0^1} = \left(\int_\Omega g^2 dx + \sum_{i=1}^{n} \int_\Omega g_i^2 dx \right)^{1/2}$$

which proves inequality in one direction of (8.9).

Conversely, suppose that $f \in D'(\Omega)$ is a distribution of the form (8.8). Then, using the definition of the distributional derivative, we have for any $\varphi \in D(\Omega)$ that

$$\langle f, \phi \rangle = \langle f_0, \phi \rangle + \sum_{i=1}^{n} \langle \partial_i f_i, \phi \rangle = \langle f_0, \phi \rangle - \sum_{i=1}^{n} \langle f_i, \partial_i \phi \rangle.$$

Use of the Cauchy-Schwartz inequality gives

$$\left|\langle f_0,\phi\rangle\right| \le \left(\langle f_0,\phi\rangle^2 + \sum_{i=1}^{n}\langle f_i,\partial_i\phi\rangle^2\right)^{1/2}$$

Moreover, since the fi are regular distributions belonging to $L^2(\Omega)$

$$\left|\langle f_i,\partial_i\phi\rangle\right| = \left|\int_\Omega f_i\partial_i\phi dx\right| \le \left(\int_\Omega f_i^2 dx\right)^{1/2}\left(\int_\Omega \partial_i\phi^2 dx\right)^{1/2},$$

So $$\left|\langle f,\phi\rangle\right| \le \left[\left(\int_\Omega f_0^2 dx\right)\left(\int_\Omega \phi^2 dx\right) + \sum_{i=1}^{n}\left(\int_\Omega f_i^2 dx\right)\left(\int_\Omega \partial_i\phi^2 dx\right)\right]^{1/2}$$

And

$$\left|\langle f,\phi\rangle\right| \le \left(\int_\Omega f_0^2 dx + \sum_{i=1}^{n}\int_\Omega f_i^2 dx\right)^{1/2}\left(\int_\Omega \phi^2 + \int_\Omega \partial_i\phi^2 dx\right)^{1/2} \le \left(\sum_{i=0}^{n}\int_\Omega f_i^2 dx\right)^{1/2} \|\phi\|_{H_0^1}$$

Thus the distribution $f : D(\Omega) \to R$ is bounded with respect to the $H_0^1(\Omega)$-norm on the dense subset $D(\Omega)$. It therefore extends in a unique way to a bounded linear functional on $H_0^1(\Omega)$, which we still denote by f. Moreover,

$$\| f \|_{H^{-1}} \le \left(\sum_{i=0}^{n}\int_\Omega f_i^2 dx\right)^{1/2}$$

which proves inequality in the other direction of (8.9).

The dual space of $H^1(\Omega)$ cannot be identified with a space of distributions on Ω because $D(\Omega)$ is not a dense subspace. Any linear functional $f \in H^1(\Omega)^*$ defines a distribution by restriction to $D(\Omega)$, but the same distribution arises from differentiable linear functionals. Conversely, any distribution $T \in D'(\Omega)$ that is bounded with respect to the H^1-norm extends uniquely to a bounded linear functional on $H^1\ 0$, but the extension of the functional to the orthogonal complement $\left(H^1 0\right)^\perp$ in H^1 is arbitrary (subject to maintaining its boundedness). Roughly speaking, distributions are defined on functions whose boundary values or trace is zero, but general linear functionals on H^1 depend on the trace of the function on the boundary $\partial\Omega$.

Notes

Example 8.8. The one-dimensional Sobolev space $H^1(0,1)$ is embedded in the space C([0,1]) of continuous functions, since p > n for p = 2 and n = 1. In fact, according to the Sobolev embedding theorem $H^1(0,1) \to C^{0-1/2}([0,1])$, as can be seen directly from the Cauchy-Schwartz inequality:

$$|f(x) - f(y)| \le \int_y^x |f'(t)|\,dt$$

$$\le \left(\int_y^x 1dt\right)^{1/2} \left(\int_y^x |f'(t)|^2\,dt\right)^{1/2}$$

$$\le \left(\int_0^1 |f'(t)|^2\,dt\right)^{1/2} |x-y|^{1/2}$$

As usual, we identify an element of $H^1(0,1)$ with its continuous representative in C([0,1]). By the trace theorem,

$$H_0^1(0,1) = \{u \in H^1(0,1) : u(0) = 0, u(1) = 0\}$$

The orthogonal complement is

$$H_0^1(0,1)^{\perp} = \{u \in H^1(0,1) : \text{such that}\,(u,\upsilon)_{H^1} = 0\,\text{for every}\,\upsilon \in H_0^1(0,1)\}.$$

This condition implies that $u \in H_0^1(0,1)^{\perp}$ if and only if

$\int_0^1 (u\upsilon + u'\upsilon')dx = 0$ for all $\upsilon \in H_0^1(0,1)$

which means that u is a weak solution of the ODE

$$-u'' + u = 0.$$

It follows that $u(x) = c_1 c^x + c_2 e^{-x}$,

so $H^1(0,1) = H_0^1(0,1) \oplus E$

where E is the two dimensional subspace of $H^1(0,1)$ spanned by the orthogonal vectors $\{e^x, e^{-x}\}$ Thus,

$$H^1(0,1)^* = H^{-1}(0,1) \oplus E^*$$

If $f \in H^1(0,1)^*$ and $u = u_0 + c_1 e^x + c_2 e^{-x}$ where $u_0 \in H_0^1(0,1)$, then

$$\langle f, u\rangle = \langle f_0, u_0\rangle + a_1c_1 + a_2c_2$$

Where $f_0 \in H^{-1}(0,1)$ is the restriction of f to $H_0^1(0,1)$ and

$a_1 = \langle f, e^x\rangle$, $a_2 = \langle f, e^{-x}\rangle$

The constants a_1, a_2 determine how the functional $f \in H^1(0,1)^*$ acts on the boundary values u(0), u(1) of a function $u \in H^1(0,1)$.

8.5 THE POINCARE INEQUALITY FOR $H_0^1(\Omega)$

We cannot, in general, estimate a norm of a function in terms of a norm of its derivative since constant functions have zero derivative. Such estimates are possible if we add an additional condition that eliminates non-zero constant functions. For example, we can require that the function vanishes on the boundary of a domain, or that it has zero mean. We typically also need some sort of boundedness condition on the domain of the function, since even if a function vanishes at some point we cannot expect to estimate the size of a function over arbitrarily large distances by the size of its derivative. The resulting inequalities are called Poincare inequalities.

The inequality we prove here is a basic example of a Poincare inequality. We say that an open set Ω in R^n is bounded in some direction if there is a unit vector $e \in R^n$ and constants a, b such that $a < x.e < b$ for all $x \in \Omega$.

THEOREM 8.9.

Suppose that Ω is an open set in R^n that is bounded is some direction. Then there is a constant C such that

(8.11) $$\int_\Omega u^2 dx \le C\int_\Omega |Du|^2 dx \text{ for all } u \in H_0^1(\Omega)$$

PROOF. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it is sufficient to prove the inequality for $u \in C_c^\infty(\Omega)$.

Notes

The inequality is invariant under rotations and translations, so we can assume without loss of generality that the domain is bounded in the x_{n-} direction and lies between $0 < x_n < a$.

Writing $x = (x^1, x_n)$ where $x^1 = (x_1, \ldots\ldots,, x_{n-1})$, we have

$$\left|u(x^1, x_n)\right| = \left|\int_0^{x_n} \partial_n u(x', t)\,dt\right| \le \int_0^a \left|\partial_n u(x', t)\right| dt\ .$$

The Cauchy-Schwartz inequality implies that

$$\int_0 a\left|\partial_n u(x', t)\right| dt = \int_0^a 1.\left|\partial_n u(x', t)\right| dt \le a^{1/2}\left(\int_0^a \left|\partial_n u(x', t)\right|^2 dt\right)^{1/2}$$

Hence,

$$\left|u(x', x_n)\right|^2 \le a\int_0^n \left|\partial_n u(x', t)\right|^2 dt$$

Integrating this inequality with respect to x_n, we get

$$\int_0^a \left|u(x', x_n)\right|^2 dx_n \le a^2 \int_0^a \left|\partial_n u(x', t)\right|^2 dt$$

A further integration with respect to x' gives

$$\int_\Omega \left|u(x)\right|^2 dx \le a^2 \int_\Omega \left|\partial_n u(x)\right|^2 dx.$$

Since $\left|\partial_n u\right| \le \left|Du\right|$, the result follows with $C = a^2$.

Check your progress

1. Prove: Suppose that Ω is an open set in Rn that is bounded is some direction. Then there is a constant C such that for all .

8.6 EXISTENCE OF WEAK SOLUTIONS OF THE DIRICHLET PROBLEM

This inequality implies that we may use as an equivalent inner-product on H_0^1 an expression that involves only the derivatives of the functions and not the functions themselves.

Corollary 8.10. If Ω is an open set that is bounded in some direction, then $H_0^1(\Omega)$ equipped with the inner product

$$(8.12) \quad (u,\upsilon)_0 = \int_\Omega Du \,.\, D\upsilon \, dx$$

is a Hilbert space, and the corresponding norm is equivalent to the standard norm on $H_0^1(\Omega)$.

PROOF. We denote the norm associated with the inner-product (8.12) by

$$\|u\|_0 = \left(\int_\Omega |Du|^2 \, dx\right)^{1/2}$$

And the standard norm and inner product by

$$(8.13) \quad \|u\|_1 = \left(\int_\Omega \left[u^2 + |Du|^2\right] dx\right)^{1/2}$$

$$(u,\upsilon)_1 = \int_\Omega (u\upsilon + Du.D\upsilon) \, dx$$

Then, using the Poincare inequality (8.11), we have

$$\|u\|_0 \le \|u\|_1 \le (C+1)^{1/2} \|u\|_0 .$$

Thus, the two norms are equivalent; in particular, $\left(H_0^1, (.,.)_0\right)$ is complete since $\left(H_0^1, (.,.)_1\right)$ is complete, so it is a Hilbert space with respect to the inner product (8.12).

Existence of weak solutions of the Dirichlet problem with these preparations, the existence of weak solutions is an immediate consequence of the Riesz representation theorem.

THEOREM 8.11. Suppose that Ω is an open set in R^n that is bounded in some direction and $f \in H^{-1}(\Omega)$. Then there is a unique weak solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ in the sense of Definition 8.2.

PROOF. Proof. We equip $H_0^1(\Omega)$ with the inner product (8.12). Then, since **Ω** is bounded in some direction, the resulting norm is equivalent to the standard norm, and f is a bounded linear functional on $\left(H_0^1(\Omega),(,)_0\right)$. By the Riesz representation theorem, there exists a unique $u \in H_0^1(\Omega)$ such that

$$(u,\phi)_0 = \langle f,\phi\rangle \text{ for all } u \in H_0^1(\Omega)$$

Which is equivalent to the condition that u is a weak solution.

The same approach works for other symmetric linear elliptic PDEs. Let us give some examples.

EXAMPLE 8.12. Consider the Dirichlet problem

$$-\Delta u + u = f \text{ in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

Then $u \in H_0^1(\Omega)$ is a weak solution if

$$\int_\Omega (Du, D\phi + u\phi)\,dx = \langle f,\phi\rangle \text{ for all } \phi \in H_0^1(\Omega)$$

This is equivalent to the condition that

$$(u,\phi)_1 = \langle f,\phi\rangle \text{ for all } \phi \in H_0^1(\Omega)$$

where (·,·)1 is the standard inner product on $H_0^1(\Omega)$) given in (8.13). Thus, the Riesz representation theorem implies the existence of a unique weak solution. Note that in this example and the next, we do not use the Poincare inequality, so the result applies to arbitrary open sets, including $\Omega = R^n$. In that case, $H_0^1(R^n) = H^1(R^n)$, and we get a unique solution $u \in H^1(R^n)$ of $-\Delta u + u = f$ for every $f \in H^{-1}(R^n)$. Moreover, using the standard norms, we have $\|u\|_{H^1} = \|f\|_{H^{-1}}$. Thus the operator $-\Delta + I$ is an isometry of $H^1(R^n)$ onto $H^{-1}(R^n)$.

Example 8.13. As a slight generalization of the previous example, suppose that μ > 0. A $u \in H_0^1(\Omega)$ is a weak solution of

(8.14) $$-\Delta u + \mu u = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

If $(u,\phi)_\mu = \langle f, \phi\rangle$ for all $\phi \in H_0^1(\Omega)$ where

$$(u,\upsilon)_\mu = \int_\Omega (\mu u\upsilon + Du.D\upsilon)\,dx$$

The norm $\|.\|_\mu$ associated with this inner product is equivalent to the standard one, since

$$\frac{1}{C}\|u\|_\mu^2 \le \|u\|_1^2 \le C\|u\|_\mu^2$$

Where $C = \max\left\{\mu, \frac{1}{\mu}\right\}$. We therefore again get the existence of a unique weak solution from the Riesz representation theorem.

Example 8.14.

Consider the last example for $\mu < 0$. If we have a Poincare inequality $\|u\|_{L^2} \le C\|Du\|_{L^2}$ for Ω, which is the case if Ω is bounded in some direction, then

$$(u,u)_\mu = \int_\Omega \left(\mu u^2 + |Du|^2\right)dx \ge (1 - C|\mu|)\int_\Omega |Du|^2\,dx$$

Thus $\|u\|_\mu$ defines a norm on $H_0^1(\Omega)$ that is equivalent to the standard norm if $\frac{-1}{C} < \mu < 0$, and we get a unique weak solution in this case also, provided that $|\mu|$ is sufficiently small.

For bounded domains, the Dirichlet Laplacian has an infinite sequence of real eigenvalues $\{\lambda_n : n \in R\}$ such that there exists a nonzero solution $u \in H_0^1(\Omega)$ of $-\Delta u = \lambda_n u$. The best constant in the Poincare inequality can be shown to be the minimum eigenvalue λ_1, and this method does not work if $\mu \le -\lambda_1$. For $\mu = -\lambda_n$, a weak solution of (8.14) does not exist for every $f \in H^{-1}(\Omega)$, and if one does exist it is not unique since we can add to it an arbitrary eigenfunction. Thus, not only does the method fail, but the conclusion of Theorem 8.11 may be false.

Check your progress

2. Explain about existence of week solutions.

8.7 GENERAL LINEAR, SECOND ORDER ELLIPTIC PDES

Example 8.15. Consider the second order PDE

(8.15) $$-\sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_i u \right) = f \text{ in } \Omega$$

where the coefficient functions $a_{ij} : \Omega \to R$ are symmetric $\left(a_{ij} = a_{ij} \right)$, bounded, and satisfy the uniform ellipticity condition that for some $\theta > 0$

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \text{ for all } x \in \Omega \text{ and all } \xi \in R^n.$$

Also, assume that Ω is bounded in some direction. Then a weak formulation of (8.15) is that $u \in H_0^1(\Omega)$ and

$$a(u,\phi) = \langle f, \phi \rangle \text{ for all } \phi \in H_0^1(\Omega)$$

Where the symmetric bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to R$ is defined by

$$a(u,\upsilon) = \sum_{i,j=1}^{n} \int a_{ij} \partial_i u \, \partial_j \upsilon \, dx.$$

The boundedness of a_{ij}, the uniform ellipticity condition, and the Poincare inequality imply that a defines an inner product on H_0^1 which is equivalent to the standard one. An application of the Riesz representation theorem for the bounded linear functionals f on the Hilbert space $\left(H_0^1, a \right)$ then implies the existence of a unique weak solution. We discuss a generalization of this example in greater detail in the next section.

General linear, second order elliptic PDEs

Consider PDEs of the form

$$Lu = f$$

where L is a linear differential operator of the form

$$\text{(8.16)} \qquad Lu = -\sum_{i,j=1}^{n} \partial_i\left(a_{ij}\partial_j u\right) + \sum_{i=1}^{n} \partial_i\left(b_i u\right) + cu$$

acting on functions $u : \Omega \to \mathbb{R}$ where Ω is an open set in Rn. A physical interpretation of such PDEs is described briefly in Section 8.A.

We assume that the given coefficients functions $a_{ij}, b_i, c : \Omega \to R$ satisfy

$$\text{(8.17)} \qquad a_{ij}, b_i, c \in L^{\infty}\Omega,\ a_{ij} = a_{ji}.$$

The operator L is elliptic if the matrix (aij) is positive definite. We will assume the stronger condition of uniformly ellipticity given in the next definition.

Definition 8.16. The operator L in (8.16) is uniformly elliptic on Ω if there exists a constant θ > 0 such that

$$\text{(8.18)} \qquad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

For x almost every where in Ω and every $\xi \in R^n$.

This uniform ellipticity condition allows us to estimate the integral of $|Du|^2$ in terms of the integral of $\sum a_{ij}\partial_i u\, \partial_j u$.

Example 8.17. The Laplacian operator $L = -\Delta$ is uniformly elliptic on any open set, with θ = 1.

Example 8.18. The Tricomi operator

$$L = y\partial_x^2 + \partial_y^2$$

is elliptic in y>0 and hyperbolic in y<0. For any $0 < \in < 1$, L is uniformly elliptic in the strip $\{(x, y) : \in < y < 1\}$, with $\theta = \in$, but it is not uniformly elliptic in $\{(x, y) : 0 < y < 1\}$.

Notes

For $\mu \in R$, we consider the Dirichlet problem for $L + \mu I$,

(8.19) $$Lu + \mu u = f \text{ in } \Omega$$

$$u=0 \quad \text{on } \partial\Omega$$

We motivate the definition of a weak solution of (8.19) in a similar way to the motivation for the Laplacian: multiply the PDE by a test function $\phi \in C_c^\infty(\Omega)$, integrate over Ω, and use integration by parts, assuming that all functions and the domain are smooth. Note that

$$\int_\Omega \partial_i (b_i u)\phi dx = -\int_\Omega b_i u \partial_i \phi dx.$$

This leads to the condition that $u \in H_0^1(\Omega)$ is a weak solution of (8.19) with L given by (8.16) if

$$\int_\Omega \left\{ \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j \phi - \sum_{i=1}^n b_i u \partial_i \phi + cu\phi \right\} dx + u\int_\Omega u\phi dx = \langle f, \phi \rangle$$

For all $\phi \in H_0^1(\Omega)$.

To write this condition more concisely, we define a bilinear form

$a : H_0^1(\Omega) \times H_0^1(\Omega) \to R$ by

(8.20) $$a(u, \upsilon) = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij}\partial_i u \partial_i \upsilon - \sum_{i}^n b_i u \partial_i \upsilon + cu\upsilon \right\} dx.$$

Definition 8.19. Suppose that Ω is an open set in $R^n, f \in H^{-1}(\Omega)$, and L is a differential operator (8.16) whose coefficients satisfy (8.17). Then $u : \Omega \to R$ is a weak solution of (8.19) if: $(a) u \in H_0^1(\Omega); (b)$

$$a(u, \phi) + \mu (u, \phi)_{L^2} = \langle f, \phi \rangle \text{ for all } \phi \in H_0^1(\Omega).$$

The form a in (8.20) is not symmetric unless $b_i = 0$. We have

$$a(\upsilon, u) = a^*(u, \upsilon)$$

Where

(8.21) $$a^*(u,\upsilon)=\int_\Omega\left\{\sum_{i,j=1}^{n}a_{ij}\partial_i u\partial_j\upsilon+\sum_{i}^{n}b_i\left(\partial_i u\right)\upsilon+cu\upsilon\right\}$$

is the bilinear form associated with the formal adjoint L∗ of L,

(8.22) $$L^*u=-\sum_{i,j=1}^{n}\partial_i\left(a_{ij}\partial_i u\right)-\sum_{i=1}^{n}b_i\partial_i u+cu\,.$$

The proof of the existence of a weak solution of (8.19) is similar to the proof for the Dirichlet Laplacian, with one exception. If L is not symmetric, we cannot use a to define an equivalent inner product on $H_0^1(\Omega)$ and appeal to the Riesz representation theorem. Instead we use a result due to Lax and Milgram which applies to non-symmetric bilinear forms.3

Check your progress

3. Discuss about general linear, second order elliptic PDEs.

8.8 LET US SUM UP

In this unit we have discussed about weak formulation of the Dirichlet problem, Variation formulation, The space, The Poincare inequality, Existence of weak solutions of the Dirichlet problem, General linear, second order elliptic PDEs. Of a weak solution in is closely connected with the variational formulation of the Dirichlet problem for Poisson's equation. The negative order Sobolev space $H^{-1}(\Omega)$ can be described as a space of distributions on **Ω**. If **Ω** is an open set that is bounded in some direction, then $H_0^1(\Omega)$ equipped with the inner product $(u,\upsilon)_0=\int_\Omega Du\,.\,D\upsilon\;dx$ is a Hilbert space, and the corresponding norm is equivalent to the standard norm on $H_0^1(\Omega)$.

8.9 KEY WORDS

Notes

1.Dirichlet problem for the Laplacian with homogeneous boundary conditions.

2.Of a weak solution in is closely connected with the variational formulation of the Dirichlet problem for Poisson's equation.

3.The space $H^{-1}(\Omega)$ consists of all distributions $f \in D'(\Omega)$ of the form

$f = f_0 + \sum_{i=1}^{n} \partial_i f_i$ where $f_0, f_i \in L^2(\Omega)$

4.If Ω is an open set that is bounded in some direction, then $H_0^1(\Omega)$ equipped with the inner product.

5.The Laplacian operator $L = -\Delta$ is uniformly elliptic on any open set, with $\theta = 1$.

8.10 QUESTIONS FOR REVIEW

1.Discuss about weak formulation of the Dirichlet problem

2.Discuss about variation formulation

3.Discuss about Poincare inequality

4.Discuss about general linear, second order elliptic PDEs

8.11 SUGGESTED READINGS AND REFERENCES

1. S. L. Ross, Differential Equations, 3rd Edn., Wiley India, 1984.
2. DiBenedetto, Partial Differential Equations, Birkhaüser, 1995.
3. L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.
4. I.N. Sneddon Elements of Partial Differential Equations McGrawHill 1986.
5. R. Churchil & J. Brown, Fourier Series & Boundary Value Problems.
6. R.C. McOwen , Partial Differential Equations (Pearson Edu.) 2003.
7. Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum, Outline Series, McGraw hill Series.
8. Partial Differential Equations, -Walter A.Strauss
9. Partial Differential Equations,-John K.Hunter
10. Partial Differential Equations,*Erich Mieremann*

11. Partial Differential Equations,-*Victor Ivrii*

8.12 ANSWERS TO CHECK YOUR PROGRESS

1. See section 8.5

2. See section 8.6

3. See section 8.7

UNIT-9 THE LAX-MILGRAM THEOREM AND GENERAL ELLIPTIC PDES

STRUTURE

9.0 OBJECTIVE

In this unit we will learn about The Lax-Milgram theorem, Compactness of the resolvent, The Fredholm alternative, The spectrum of a self – adjoint elliptic operator, Interior regularity, Boundary regularity, Some further perspectives.

9.1 INTRODUCTION

We begin by stating the Lax-Milgram theorem for a bilinear form on a Hilbert space. Afterwards, we verify its hypotheses for the bilinear form associated with a general second-order uniformly elliptic PDE and use it to prove the existence of weak solutions.

9.2 THE LAX-MILGRAM THEOREM

THEOREM 9.1. Let H be Hilbert space with inner-product $(.,.):\mathrm{H}\times\mathrm{H}\to\mathrm{R}$, and let $\mathrm{a}:\mathrm{H}\times\mathrm{H}\to\mathrm{R}$, be a bilinear form on H. Assume that there exist constants $\mathrm{C}_1,\mathrm{C}_2>0$ such that

$$C_1\|u\|^2\leq a(u,u),|a(u,v)| \text{ for all } u,v\in H$$

Then for every bounded linear functional $f:H\to R$, there exists a unique $u\in H$ such that

$$\langle f,v\rangle=a(u,v) \ \textit{for all } v\in H$$

The verification of the hypotheses for (9.1) depends on the following energy estimates.

THEOREM 9.2. Let a be the bilinear form on $H_0^1(\Omega)$ defined in (9.1), where the coefficients satisfy (8.17) and the uniform ellipticity condition (8.18) with constant θ. Then there exist constant $C_1,C_2>0$ and $\gamma\in R$ such that for all $u,v\in H_0^1(\Omega)$

(9.4) $$C_1\|u\|^2_{H_0^1}\leq a(u,u)+\gamma\|u\|^2_{L^2}$$

(9.4) $$|a(u,v)|\leq C_2\|u\|_{H_0^1}\|v\|_{H_0^1}$$

If b=0, we may take $\gamma=\theta-c_0$ where $c_0=\inf_\Omega c,$ *and if* $b\neq 0$, we may take

$$\gamma=\frac{1}{2\theta}\sum_{i=1}^{n}\|b_i\|^2_{L^\infty}+\frac{\theta}{2}-c_0$$

PROOF. First, we have for any $u,v\in H_0^1(\Omega)$ that

$$\left|a(u,v)\right| \le \sum_{i,j=1}^{n} \int_\Omega \left|a_{ij}\partial_i u \partial_i v\right| dx + \sum_{i=1}^{n} \int_\Omega \left|b_i u \partial_i v\right| dx + \int_\Omega \left|cuv\right| dx.$$

$$\le \sum_{i,j=1}^{n} \left\|a_{ij}\right\|_{L^\infty} \left\|\partial_i u\right\|_{L^2} \left\|\partial_j v\right\|_{L^2}$$

$$+\sum_{i=1}^{n} \left\|a_{ij}\right\|_{L^\infty} \left\|u\right\|_{L^2} \left\|\partial_i v\right\|_{L^2} + \left\|c\right\|_{L^\infty} \left\|u\right\|_{L^2} \left\|v\right\|_{L^2}$$

$$\le C\left(\sum_{i,j=1}^{n} \left\|a_{ij}\right\|_{L^\infty} + \sum_{i=1}^{n} \left\|b_i\right\|_{L^\infty} + \left\|c\right\|_{L^\infty}\right) \left\|u\right\|_{H_0^1} \left\|v\right\|_{H_0^1}$$

Which shows (9.5).

Second, using the uniform ellipticity condition (4.18), we have

$$\theta \left\|Du\right\|_{L^2}^2 = \theta \int_\Omega \left|Du\right|^2 dx$$

$$\le \sum_{i,j=1}^{n} \int_\Omega a_{ij}\partial_i u \partial_j u dx$$

$$\le a(u,v) + \sum_{i=1}^{n} \int_\Omega b_i u\, \partial_i u dx - \int_\Omega cu^2 dx$$

$$\le a(u,u) + \sum_{i=1}^{n} \int_\Omega \left|b_i u\, \partial_i u\right| dx - c_0 \int_\Omega u^2 dx$$

$$\le a(u,v) + \sum_{i=1}^{n} \left\|b_i\right\|_{L^\infty} \left\|u\right\|_{L^2} \left\|\partial_i u\right\|_{L^2} - c_0 \left\|u\right\|_{L^2}$$

$$\le a(u,u) + \beta \left\|u\right\|_{L^2} \left\|Du\right\|_{L^2} - c_0 \left\|u\right\|_{L^2},$$

Where $c(x) \ge c_0 a.e$. In Ω, and

$$\beta = \left(\sum_{i=1}^{n} \left\|b_i\right\|_{L^\infty}^2\right)^{1/2}$$

If $\beta = 0$, we get (9.4) with

$$\gamma = \theta - c_0, \quad C_1 = \theta$$

If $\beta > 0$, by Cauchy's inequality with $\in$, we have for any $\in > 0$ that

$$\|u\|_{L^2}\|Du\|_{L^2} \le \in \|Du\|_{L^2}^2 + \frac{1}{4\in}\|u\|_{L^2}^2 .$$

Hence, choosing $\in = \dfrac{\theta}{2\beta}$, we get

$$\frac{\theta}{2}\|Du\|_{L^2}^2 \le a(u,u) + \left(\frac{\beta^2}{2\theta} - c_0\right)\|u\|_{L^2},$$

And (9.4) follows with

$$\gamma = \frac{\beta^2}{2\theta} + \frac{\theta}{2} - c_0, \quad C_1 = \frac{\theta}{2}$$

Check your progress

1. Explain about the Lax-Milgram theorem

9.3 COMPACTNESS OF THE RESOLVENT

Equation (9.4) is called Garding's inequality; this estimate of the H_0^1-norm of u in terms of a(u,u), using the uniform ellipticity of L, is the crucial energy estimate. Equation (9.5) states that the bilinear form a is bounded on H_0^1. The expression for γ in this Theorem is not necessarily sharp. For example, as in the case of the Laplacian, the use of Poincare's inequality gives smaller values of γ for bounded domains.

Theorem 9.3. Suppose that Ω is an open set in R^n, and f ∈ $H^{-1}(\Omega)$. Let L be a differential operator (4.16) with coefficients that satisfy (4.17), and let $\gamma \in R$ be a constant for which Theorem 9.2 holds. Then for every $\mu \ge \gamma$ there is a unique weak solution of the Dirichlet problem

$$Lu + \mu f = 0, \quad u \in H_0^1(\Omega)$$

In the sense of Definition 8.19

PROOF. For $\mu \in R$, define $a_\mu : H_0^1(\Omega) \times H_0^1(\Omega) \to R$ by

(9.6) $$a_\mu(u,v)=a(u,v)+\mu(u,v)_{L^2}$$

Where a is defined in (9.1). Then $u\in H_0^1(\Omega)$ is weak solution of $Lu+\mu u=f$ if and only if

$a_\mu(u,\phi)=\langle f,\phi\rangle$ for all $\phi\in H_0^1(\Omega)$

From (9.5)

$$|a_\mu(u,v)|\le C_2\|u\|_{H_0^1}+|u|\|u\|_{L^2}\le(C_2+|\mu|)\|u\|_{H_0^1}\|v\|_{H_0^1}$$

So, a_μ is bounded on $H_0^1(\Omega)$. From (9.4)

$$C_1\|u\|_{H_0^1}^2\le a(u,u)+\gamma\|u\|_{L^2}^2\le a_\mu(u,u)$$

Where $\mu\ge\gamma$. Thus, by the Lax-Milgram theorem, for every $f\in H^{-1}(\Omega)$ there is a unique $u\in H_0^1(\Omega)$ such that $\langle f,\phi\rangle=a_\mu(u,\phi)$ for all $v\in H_0^1(\Omega)$, which proves the result.

Although L* is not of exactly the same form as L, since it first derivative term is not in divergence form, the same proof of the existence of weak solutions for L applies to L* with a in (9.1) replaced by a∗ in (9.2).

An elliptic operator L + μI of the type studied above is a bounded, invertible linear map from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ for sufficiently large $\mu\in R$, so we may de-fine an inverse operator $K=(L+\mu I)^{-1}$. If Ω is a bounded open set, then the Sobolev embedding theorem implies that $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, and therefore K is a compact operator on $L^2(\Omega)$.

The operator $(L-\lambda I)^{-1}$ is called the resolvent of L, so this property is sometimes expressed by saying that L has compact resolvent. As discussed in Example 4.14, $L+\mu I$ may fail to be invertible at smaller values of μ, such that λ = −μ belongs to the spectrum σ(L) of L, and the resolvent is not defined as a bounded operator on $L^2(\Omega)$ for λ∈ σ(L).

The compactness of the resolvent of elliptic operators on bounded open sets has several important consequences for the solvability of the elliptic PDE and the spectrum of the elliptic operator. Before describing some of these, we discuss the resolvent in more detail.

From Theorem 9.3, for $\mu \geq \gamma$ we can define

$$K : L^2(\Omega) \to L^2(\Omega), \; K = (L + \mu I)^{-1}|_{L^2(\Omega)}.$$

We define the inverse K on $L^2(\Omega)$, rather than $H^{-1}(\Omega)$, in which case its range is a subspace of $H_0^1(\Omega)$. If the domain Ω is sufficiently smooth for elliptic regularity theory to apply, then $u \in H^2(\Omega)$ if $f \in L^2(\Omega)$, and the range of K is $H^2(\Omega) \cap H_0^1(\Omega)$; for non-smooth domains, the range of K is more difficult to describe.

If we consider L as an operator acting in $L^2(\Omega)$, then the domain of L is D=ran K, and

$$L : D \subset L^2(\Omega) \to L^2(\Omega)$$

Is an unbounded linear operator with dense domain D. The operator L is closed, meaning that if $\{u_n\}$ is a sequence of functions in D such that $u_n \to u$ and $Lu_n \to f$ in $L^2(\Omega)$, then $u \in D$ and $Lu = f$. By using the resolvent, we can replace and analysis of the unbounded operator L by an analysis of the bounded operator K.

If $f \in L^2(\Omega)$, then $\langle f, v \rangle = (f, v)_{L^2}$. It follows from the definition of weak solution of $Lu + \mu u = f$ that

(9.7) $Kf = u$ if and only if $a_\mu(u, v) = (f, v)_{L^2}$ for all $v \in H_0^1(\Omega)$ where a_μ is defined in (9.6). We also define the operator

$$K^* : L^2(\Omega) \to L^2(\Omega), \; K^* = (L^* + \mu I)^{-1}|_{L^2(\Omega)},$$

Meaning that

(9.8) $K^* f = u$ if and only if $a_\mu^*(u,v) = (f,v)_{L^2}$ for all $v \in H_0^1(\Omega)$

where $a_\mu^*(u,v) = a^*(u,v) + \mu(u,v)_{L^2}$ and a^* is given in (9.2)

THEOREM 9.4. If $K \in B(L^2(\Omega))$ is defined by (9.7), then the adjoint of K is K^*defined by (9.8). If Ω is a bounded open set, then K is a compact operator.

PROOF. If $f,g \in L^2(\Omega)$ and $Kf = u, K^* g = v$, then using (9.7) and (9.8), we get,

$$(f, K^* g)_{L^2} = (f,v)_{L^2} = a_\mu(u,v) = a_\mu^*(u,v) = (g,u)_{L^2} = (u,g)_{L^2} = (Kf,g)_{L^2}$$
.

Hence, K^* is the adjoint of K

If $Kf = u$, then (9.4) with $\mu \geq \gamma$ and (9.7) imply that

$$C_1 \|u\|_{H_0^1}^2 \leq a_\mu(u,u) = (f,u)_{L^2} \leq \|f\|_{L^2} \|u\|_{L^2} \leq \|f\|_{L^2} \|u\|_{H_0^1}.$$

Hence $\|Kf\|_{H_0^1} \leq C\|f\|_{L^2}$ where $C = \dfrac{1}{C_1}$. It follows that K is compact if Ω is bounded, since it maps bounded sets in $L^2\Omega$ into bounded sets in $H_0^1\Omega$, which are pre compact in $L^2\Omega$ by the Sobolev embedding theorem.

9.4 THE FREDHOLM ALTERNATIVE

Consider the Dirichlet problem

(9.9) $$Lu = f \ \ in \ \ \Omega, \ \ u = 0 \ \ on \ \ \partial\Omega$$

Where Ω is a smooth, bounded open set, and

$$Lu = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u\right) + \sum_{i=1}^{n} \partial_i \left(b_i u\right) + cu .$$

If $u = v = 0$ on $\partial\Omega$, Green's formula implies that

$\int_\Omega (Lu) v dx = \int_\Omega u (L^* v) dx,$

Where the formal adjoint L^* of L is defined by

$$L^*v = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j v \right) - \sum_{i=1}^{n} b_i \partial_i v + cv$$

It follows that if u is a smooth solution of (9.9) and v is a smooth solution of the homogeneous adjoint problem,

$$L^*v = 0 \ \ in\ \Omega, \quad v = 0 \ \ on\ \partial\Omega$$

Then

$$\int_\Omega fvdx = \int_\Omega (Lu)vdx = \int_\Omega uL^*vdx = 0.$$

Thus, a necessary condition for (9.9) to be solvable is that f is orthogonal with respect to the $L^2(\Omega)$-inner product to every solution of the homogeneous adjoint problem.

For bounded domains, we will use the compactness of the resolvent to prove that this condition is necessary and sufficient for the existence of a weak solution of (9.9) where $f \in L^2(\Omega)$. Moreover, the solution is unique if and only if a solution exists for every $f \in L^2(\Omega)$.

This result is a consequence of the fact that if K is compact, then the operator $I+\sigma K$ is a Fredholm operator with index zero on $L^2(\Omega)$ for any $\sigma \in R$, and therefore satisfies the Fredholm alternative. Thus, if $K = (L + \mu I)^{-1}$ is compact, the inverse elliptic operator $L-\lambda I$ also satisfies the Fredholm alternative.

THEOREM 9.5. Suppose that Ω is a bounded open set in R^n and L is a uniformly elliptic operator of the form (8.16) whose coefficients satisfy (8.17). Let L* be the adjoint operator (9.3) and $\lambda \in R$. Then one of the following two alternatives holds.

(1) The only weak solution of the equation $L^*v - \lambda v = 0$ is v=0. For every $f \in L^2(\Omega)$ there is a unique weak solution $u \in H_0^1(\Omega)$ of the equation $Lu - \lambda u = f$. In particular, the only solution of $Lu - \lambda u = 0$ is u=0.

2) The equation $L^*v = \lambda v = 0$ has a nonzero weak solution v. The solution spaces of $Lu - \lambda u = 0$ and $L^* - \lambda v = 0$ are finite-

dimensional and have the same dimension. For $f \in L^2(\Omega)$, the equation $Lu - \lambda u = f$ has a weak solution $u \in H_0^1(\Omega)$ if $(f,v) = 0$ for every $v \in H_0^1(\Omega)$ such that $L^* v - \lambda v = 0$, and if a solution exists it is not unique.

PROOF. Since $K = (L + \mu I)^{-1}$ is a compact operator on $L^2(\Omega)$, the Fredholm alternative holds for the equation

(9.10) $$u + \sigma K u = g \quad u, g \in L^2(\Omega)$$

For any $\sigma \in R$. Let us consider the two alternatives separately.

First, suppose that the only solution of $v + \sigma K^* v = 0$ is v=0, which implies that the only solution of $L^* v + (\mu + \sigma) v = 0$ is v=0. Then the Fredholm alternative for $I + \sigma K$ implies that (9.10) has a unique solution $u \in L^2(\Omega)$ for every $g \in L^2(\Omega)$. In particular, for any $g \in ran\, K$, there exists a unique solution $u \in L^2(\Omega)$, and the equation implies that $u \in rankK$. Hence, we any apply $L + \mu I$ to (9.10),

Taking $\sigma = -(\lambda + \mu)$, we get part (1) of the Fredholm alternative for L.

Second, suppose that $v + \sigma K^* v = 0$ has a finite-dimensional subspace of solutions $v \in L^2(\Omega)$. It follows that $v \in ran\, K^*$ (clearly, $\sigma \neq 0$ in this case) and

$$L^* v + (\mu + \sigma) v = 0 .$$

By the Fredholm alternative, the equation $u + \sigma K u = 0$ has a finite-dimensional subspace of solutions of the same dimension, and hence so does

$$Lu + (\mu + \sigma) u = 0 .$$

Equation (9.10): is solvable for $u \in L^2(\Omega)$ given $g \in ran\ K$ if and only if

(9.12) $(v,g)_{L^2}=0$ for all $v\in L^2(\Omega)$ such that $v+\sigma K^*v=0$, and then $u\in ran\,K$. It follows that the condition (9.12) with $g=Kf$ is necessary and sufficient for the solvability of (9.11) given $f\in L^2(\Omega)$. Since

$$(v,g)_{L^2}=(v,Kf)_{L^2}=\left(K^*v,f\right)_{L^2}=-\frac{1}{\sigma}(v,f)_{L^2}$$

And $v+\sigma K^*v=0$ if and only if $L^*v+(\mu+\sigma)v=0,$ we conclude that (9.11) is solvable for u if and only if $f\in L^2(\Omega)$ satisfies

$$(v,f)_{L^2}=0 \text{ for all } v\in ran\,K \text{ such that } L^*v+(\mu+\sigma)v=0.$$

Taking $\sigma=-(\lambda+\mu),$ we get alternative (2) for L.

Elliptic operators on a Riemannian manifold may have nonzero Fredholm index. The Atiyah-Singer index theorem (1968) relates the Fredholm index of such operators with a topological index of the manifold.

Check your progress

1. Prove theorem 9.4

9.5 THE SPECTRUM OF A SELF-ADJOINT ELLIPTIC OPERATOR

Suppose that L is a symmetric, uniformly elliptic operator of the form

(9.13)
$$Lu=-\sum_{i,j=1}^{n}\partial_i\left(a_{ij}\partial_j u\right)+cu$$

Where $a_{ij}=a_{ij}$ and $a_{ij},c\in L^\infty(\Omega)$. The associated symmetric bilinear form

$$a:H_0^1(\Omega)\times H_0^1(\Omega)\to R$$

Notes

Is given by

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij} \partial_i u \partial_i u + cuv \right) dx.$$

The resolvent $K = (L + \mu I)^{-1}$ is a compact self-adjoint operator on $L^2(\Omega)$ for sufficiently large μ. Therefore its eigenvalues are real and its eigenfunctions provide an orthonormal basis of $L^2(\Omega)$. Since L has the same eigenfunctions as K, we get the corresponding result for L.

Theorem 9.6. The operator L has an increasing sequence of real eigenvalues of finite multiplicity

$$\lambda_1 < \lambda_2 \le \lambda_3 \le \ldots\ldots \le \lambda_n \le \ldots.$$

Such that $\lambda_n \to \infty$. There is an orthonormal basis $\{n : n \in R\}$ of $L^2(\Omega)$ consisting of eigenfunctions functions $\phi_n H_0^1(\Omega)$ such that

$$L\phi_n = \lambda_n \phi_n$$

Proof. If $K\phi = 0$ for any $\phi \in L^2(\Omega)$, then applying $L + \mu I$ to the equation we find that $\phi = 0$, so 0, is not an eigenvalue of K. If $K\phi = k\phi$, for $\phi \in L^2(\Omega)$ and $k \ne 0$, then $\phi \in ranK$ and

$$L\phi\left(\frac{1}{k} - \mu\right)\phi,$$

So ϕ is an eigenfunction of L with eigenvalue $\lambda = 1/k - \mu$. From Garding's inequality (9.4) with $u = \phi$ and the fact that $a(\phi,\phi) =; \lambda \|\phi\|_{L^2}^2$, we get

$$C_1 \|\phi\|_{H_0^1}^2 \le (\lambda + \gamma) \|\phi\|_{L^2}^2 .$$

It follows that $\lambda > -\gamma$, so the eigenvalues of L are bounded from below, and at most a finite number are negative. The spectral theorem for the compact self adjoint operator K then implies the result.

The boundedness of the domain Ω is essential here, otherwise K need not be compact, and the spectrum of L need not consist only of eigenvalues.

Example 9.7.

Suppose that $\Omega = R^n$ and $L = \Delta$. Let $K = (-\Delta + 1)^{-1}$. Then, form Example 4.12 $K : L^2(R^n) \to L^2(R^n)$. The range of k is $H^2(R^n)$. This operator is bounded but not compact. For example, if $\phi C_c^\infty(R^n)$ is any nonzero function and $\{a_j\}$ is a sequence in R^n such that $|a_j| \uparrow \infty$ as $j \to \infty$, then the sequence $\{\phi_j\}$ defined by $\phi_j(x) = \phi(x - a_j)$ is bounded in $L^2(R^n)$ but $\{K\phi_j\}$ has no convergent subsequence. In this example, K has continuous spectrum $[0,1]$ on $L^2(R^n)$ and no eigenvalues. Correspondingly, $-\Delta$ has the purely continuous spectrum $[0,\infty)$. Finally, let us briefly consider the Fredholm alternative for a self-adjoint elliptic equatio from the perspective of this spectral theory. The equation

$$(9.14) \qquad Lu - \lambda u = f$$

May be solved by expansion with respect to the eigenfunctions of L. Suppose that $\{\phi_n : n \in R\}$ is an orthonormal basis of $L^2(\Omega)$ such that $L\phi_n = \lambda_n \phi_n$, where the eigenvalues λ_n are increasing and repeated according to their multiplicity. We get the following alternatives, where all series converge in $L^2(\Omega)$:

(1) If $\lambda \neq \lambda_n$ for any $n \in R$, then (9.14) has the unique solution

$$u = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{\lambda_n - \lambda} \phi_n$$

For every $f \in L^2(\Omega)$;

(2) If $\lambda = \lambda_M$ for some $M \in R$ and $\lambda_n = \lambda_M$ for $M \le n \le N$, then (9.14) has a solutions $u \in H_0^1(\Omega)$ if and only if $f \in L^2(\Omega)$ satisfies

$$\left(f,\phi_n\right)=0 \qquad for\ M\le n\le N.$$

In that case, the solutions are

$$u=\sum_{\lambda_n\neq\lambda}\frac{\left(f,\phi_n\right)}{\lambda_n-\lambda}\phi_n+\sum_{n=M}^{N}c_n\phi_n$$

Where $\{C_M,......C_N\}$ arbitrary real constants.

9.6 INTERIOR REGULARITY

Roughly speaking, solutions of elliptic PDEs are as smooth as the data allows. For boundary value problems, it is convenient to consider the regularity of the solution in the interior of the domain and neat the boundary separately. We begin by studying the interior regularity of solutions. We follow closely the presentation in [9].

To motivate the regularity theory, consider the following simple a priori estimate for the Laplacian. Suppose that $u\in C_c^{\infty}\left(R^n\right)$. Then, integrating by parts twice, we get

$$\int\left(\Delta u^2\right)dx=\sum_{i,j=1}^{n}\int\left(\partial_{ii}^2u\right)\left(\partial_{jj}^2u\right)dx$$

$$=-\sum_{i,j=1}^{n}\int\left(\partial_{iij}^3u\right)\left(\partial_j u\right)dx$$

$$=\sum_{i,j=1}^{n}\int\left(\partial_{ij}^2u\right)\left(\partial_{ij}^2u\right)dx$$

$$=\int\left|D^2u\right|^2dx.$$

Hence, if $-\Delta u=f$, then $\left\|D^2u\right\|_{L^2}=\|f\|_{L^2}^2$.

Thus, we can control the L^2-norm of all second derivatives of u by the L^2-norm of the Laplacian of u. This estimate suggests that we should have $u\in H_{loc}^2$ if $f,u\in L^2$ as is in fact true. The above computation is, however, not justified for weak solutions that belong to H^1; as far as we know from the previous existence theory, such solutions may not even posses second- order weak derivatives.

We will consider a PDE

(9.15) $$Lu = f \qquad in\ \Omega$$

Where Ω is an open set in $R^n, f \in L^2(\Omega)$, and L is a uniformly elliptic of then form

(9.16) $$Lu = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right).$$

It is straight forward to extend the proof of the regularity theorem to uniformly elliptic operators that contain lower-order terms [9].

A function $u \in H^1(\Omega)$ is a weak solution of (9.16) – (9.16) if

(9.17) $$a(u,v) = (f,v) \qquad \text{for all } \ v \in H_0^1(\Omega)$$

Where the bilinear form *a* is given by

(9.18) $$a(u,v) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \partial u \partial_j v \, dx.$$

We do not impose any boundary condition on *u*, for example by requiring that $u \in H_0^1(\Omega)$, so the interior regularity theorem applies to any weak solution of (9.15).

Before stating the theorem, we illustrate the idea of the proof with a further a priori estimate. To obtain a local estimate for D^2u on a subdomain $\Omega' \subset \Omega$, we introduce a cut-off function $\eta C_C^{\infty}(\Omega)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on Ω'. We take as a test function

(9.19) $$v = -\partial_k \left(\eta^2 \partial_k u\right).$$

Note that v is given by a positive definite, symmetric operator acting on *u* of a similar form to L, which leads to the positivity of the resulting estimate for $D\partial_k u$.

Notes

Multiplying (9.15) by *v* and integrating over Ω, we get $(Lu,v)=(f,v)$. Two integrations by parts imply that

$$(Lu,v)=\sum_{i,j=1}^{n}\int_{\Omega}\partial_j\left(a_{ij}\partial_i u\right)\left(\partial_k\eta^2\partial_k u\right)dx$$

$$=\sum_{i,j=1}^{n}\int\partial_k\left(a_{ij}\partial_i u\right)\left(\partial_j\partial_k u\right)dx$$

$$=\sum_{i,j=1}^{n}\int\eta^2 a_{ij}\left(\partial_i\partial_j u\right)\left(\partial_j\partial_k u\right)dx+F$$

Where

$$F=\sum_{i,j=1}^{n}\int_{\Omega}\left\{\eta^2\left(\partial_k a_{ij}\right)\left(\partial_i u\right)\left(\partial_j\partial_k u\right)\right.$$

$$\left.+2\eta\partial_j\eta\left[a_{ij}\left(\partial_i\partial_k u\right)+\left(\partial_k a_{ij}\right)\left(\partial_i u\right)\left(\partial_k u\right)\right\}\right]dx.$$

The term F is linear in the second derivatives of *u*. We use the uniform elliptically of L to get

$$\theta\int_{\Omega'}\left|D\partial_k u\right|^2 dx\le\sum_{i,j=1}^{n}\int_{\Omega}\eta^2 a_{ij}\left(\partial_i\partial_k u\right)\left(\partial_j\partial_i u\right)dx=(f,v)-F,$$

and a Cauchy inequality with $\in$ to absorb the linear terms in second derivatives on the right-hand side into the quadratic terms on the left-hand side. This results in an estimate of the form

$$\|D\partial_k u\|^2_{L^2(\Omega)}\le C\left(\|f\|^2_{L^2(\Omega)}+\|u\|^2_{H^1(\Omega)}\right)$$

The proof of regularity is entirely analogous, with the derivatives in the test function (9.19) replaced by difference quotients (see Section 4.C). We obtain and $L^2(\Omega')$-bound for the difference quotients $D\partial_l^h u$ that is uniform in *h*, which implies that $u\in H^2(\Omega')$

THEOREM 9.8. Suppose that Ω is an open set in R^n.Assume that $a_{ij}\in C^1(\Omega)$ and $f\in L^2(\Omega)$. If $u\in H^1(\Omega)$ is a weak solution of (9.15) – (9.15) – (9.16), then $u\in H^2(\Omega')$ for every $\Omega'\subset\Omega$. Furthermore, (9.20) $$\| u \|_{H^2(\Omega')} \le C\left(\| f \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)}\right)$$

Where the constant C depends only on n, Ω', Ω and a_{ij}.

PROOF. Choose a cut -off function $\eta \in C_c^\infty(\Omega)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on Ω'. We use the compactly supported test function

$$v = -D_k^{-h}\left(\eta^2 D_k^h u\right) \in H_0^1(\Omega)$$

In the definition (9.17) – (9.18) for weak solutions. (As in (9.19), v is given by a positive self-adjoint operator acting on u.) This complies that

(9.21) $$-\sum_{i,j=1}^{n} \int_\Omega a_{ij}\left(\partial_i u\right) D_k^{-h} \partial_j \left(\eta^2 D_k^h u\right) dx = -\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right) dx.$$

Performing a discrete integration by parts and using the product rule, we may write the left-hand side of (9.21) as

(9.22)

$$-\sum_{i,j=1}^{n} a_{ij}\left(\partial_i u\right) D_k^{-h} \partial_j \left(\eta^2 D_k^h u\right) dx = \sum_{i,j=1}^{n} \int_\Omega D_k^h \left(a_{ij}\partial\right) \partial_j \left(\eta^2 D_k^h u\right) dx$$

$$= \sum_{i,j=1}^{n} \int_\Omega \eta^2 a_{ij}^h \left(D_k^h \partial_i u\right)\left(D_k^h \partial_j u\right) dx + F,$$

With $a_{ij}^h(x) = a_{ij}\left(x + he_k\right)$, where the error-term F is given by

$$F = \sum_{i,j=1}^{n} \int_\Omega \left\{ \eta^2 \left(D_k^h a_{ij}\right)\left(\partial_i u\right)\left(D_k^h \partial_j u\right)\right.$$

(9.23)

$$\left. + 2\eta \partial_j \eta \left[a_{ij}^h \left(D_k^h \partial_i u\right)\left(D_k^h u\right) + \left(D_k^h a_{ij}\right)\left(\partial_i u\right)\left(D_k^h u\right)\right]\right\} dx.$$

Using the uniform ellipticity of L in (4.18), we estimate

$$\theta \int_\Omega \eta^2 \left|D_k^h D u\right|^2 dx \le \sum_{i,j=1}^{n} \int_\Omega \eta^2 a_{ij}^h \left(D_k^h \partial_i u\right)\left(D_k^h \partial_j u\right) dx.$$

Using (9.21) – (9.22) and this inequality, we find that

(9.24) $$\theta\int_\Omega \eta^2 \left|D_k^h Du\right|^2 dx \le -\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right)dx - F.$$

By the Cauchy-Schwartz inequality,

$$\left|\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right)dx\right| \le \| f \|_{L^2(\Omega)} \| D_k^{-h}\left(\eta^2 D_k^h u\right)\|_{L^2(\Omega)}.$$

Since supp $\eta \subset \Omega$, Theorem 9.53 implies that for sufficiently small *h*,

$$\left\|D_k^{-h}\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)} \le \left\|\partial_k\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)}$$

$$\le \left\|\eta^2 \partial_k D_k^h u\right\|_{L^2(\Omega)} + \left\|2\eta\left(\partial_k \eta\right) D_k^h u\right\|_{L^2(\Omega)}.$$

$$\le \left\|\eta \partial_k D_k^h u\right\|_{L^2(\Omega)} + C\left\|Du\right\|_{L^2(\Omega)}.$$

A similar estimate of F in (9.23) gives

$$|F| \le C\left(\|Du\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}^2\right).$$

Using these results in (9.24), we find that

$$\theta\left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 \le C\left(\| f \|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \| f \|_{L^2(\Omega)}\|Du\|_{L^2(\Omega)}\right.$$

$$\left.+\|Du\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}^2\right)$$

By Cauchy's inequality with $\in$, we have

$$\| f \|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} \le \in \left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 + \frac{1}{4\in}\| f \|_{L^2(\Omega)}^2$$

$$\| Du \|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} \le \in \left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 + \frac{1}{4\in}\| Du \|_{L^2(\Omega)}^2 .$$

Hence, choosing $\in$ so that $4C_\in = \theta$, and using the result in (9.25) we get that

$$\frac{\theta}{4}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)}^2 \le C\left(\| f \|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2\right).$$

Thus, since $\eta = 1$ on Ω'

(9.26) $$\left\|D_k^h Du\right\|_{L^2(\Omega')}^2 \le c\left(\| f \|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2\right)$$

Where the constant C depends on Ω, Ω', a_{ij}, but is independent of *h, u, f.* the Theorem 4.53 now implies that the weak second derivatives of *u* exist and belong to $L^2(\Omega)$. Furthermore, the $H^2 - norm$ of *u* satisfies

$$\|u\|_{H^2(\Omega')} \le C\left(\|f\|_{L^2(\Omega)} + \|u\|_{H_1(\Omega)}\right).$$

Finally, we replace $\|u\|_{H^1(\Omega)}$ in this estimate by $\|u\|_{L^2(\Omega)}$. First, by the previous argument, if $\Omega' \subset \Omega'' \subset \Omega$, then

(9.27) $$\|u\|_{H^2(\Omega)} \le C\left(\|f\|_{L^2(\Omega'')} + \|u\|_{H^1(\Omega'')}\right).$$

Let $\eta \in C_c^\infty(\Omega)$ be a cut-off function with $0 \le \eta \le 1$ and $\eta = 1$ on Ω''.

Using the uniform ellipticity of L and taking $v = \eta^2 u$ in(9.17) – (9.18), we get that

$$\theta \int_\Omega \eta^2 |Du|^2 \, dx \le \sum_{i,j=1}^n \int_\Omega \eta^2 a_{ij} \partial_i u \partial_j u \, dx$$

$$\le \int_\Omega \eta^2 fu dx - \sum_{i,j=1}^n \int_\Omega 2a_{ij} \eta u \partial_i u \partial_j \eta dx$$

$$\le \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \|Cu\|_{L^2(\Omega)} \|\eta Du\|_{L^2(\Omega)}.$$

Cauchy’s inequality with $\in$ then implies that

And since $\|\eta Du\|^2_{L^2(\Omega)} \le C\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right)$,

$\|Du\|^2_{L^2(\Omega'')} \le \|\eta Du\|^2_{L^2(\Omega)}$, the use of this result in (9.27) gives (9.20)

If $u \in H^2_{loc}(\Omega)$ and $f \in L^2(\Omega)$, then the equation $Lu = f$ relating the weak derivate of *u* and *f* holds pointwise a.e.; such solutions are often called strong solutions, to distinguish them form weak solutions, which may not possess weak second order derivatives, and classical solutions, which possess continuous second order derivatives.

The repeated application of these estimates leads to higher interior regularity.

Notes

THEOREM 9.9. Suppose that $a_{ij} \in C^{k+1}(\Omega)$ abd $f \in H^k(\Omega)$. If $u \in H^1(\Omega)$ is a weak solutions of (9.15) – (9.16), them $u \in H^{k+2}(\Omega')$ for every $\Omega' \subset \Omega$. Furthermore,

$$\|u\|_{H^{k+2}(\Omega)} \le C\left(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

Where the constant C depends only on n, k, Ω', Ω and a_{ij}.

See [9] for a detailed proof. Note that if the above conditions hold with $k > n/2$, then $f \in C(\Omega)$ and $u \in C^2(\Omega)$, so *u* is a classical solutions of the PDE $Lu = f$. Furthermore, if *f* and a_{ij} are smooth then so is the solution.

COROLLARY 9.10. If a_{ij}, $f \in C^\infty(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of

(9.15) – (9.16), then $u \in C^\infty(\Omega)$

PROOF. If $\Omega' \subset \Omega$, then $f \in H^k(\Omega')$ for evert $k \in R$, so by Theorem (9.9) $u \in H_{loc}^{k+2}(\Omega')$ for every $k \in R$, and by the Sobolev embedding theorem $u \in C^\infty(\Omega')$. Since this holds for every open set $\Omega' \subset \Omega$, we have $u \in C^\infty(\Omega)$.

9.7 BOUNDARY REGULARITY

To study the regularity of solutions near the boundary, we localize the problem to a neighborhood of a boundary point by use of a partition of unity.

We decompose the solution into a sum of functions that are compactly supported in the sets of a suitable open cover of the domain and estimate each function in the sum separately.

Assuming, as in section 1.10, that the boundary is at least C1 we may 'flatten' the boundary in a neighborhood U by a diffeomorphism $\varphi : \mathrm{U} \to \mathrm{V}$ that maps $\mathrm{U} \cap \Omega$ to an upper half space

$V = B_1(0) \cap \{y_n > 0\}$. If $\varphi^{-1} = \psi$ and $x = \psi(y)$, then by a change of variables (c.f. Theorem 1.44 and Proposition 3.21 the weak formulation (9.15) – (9.16) on U becomes

$$\sum_{i,j=1}^{n} \int_V \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial y_i} \frac{\partial \tilde{v}}{\partial y_j} dy = \int_V f \, \tilde{v} \, dy \text{ for all functions } \tilde{v} \in H_0^1(V),$$

Where $\tilde{u} \in H^1(V)$. Here, $\tilde{u} = u \ o \ \psi, \tilde{v} = v \, o \, \psi,$ and

$$\tilde{a}_{ij} = |\det D\psi| \sum_{p,q=1}^{n} a_{pq} \, o \, \psi \left(\frac{\partial \varphi_i}{\partial x_p} o \, \psi \right) \left(\frac{\partial \varphi_j}{\partial x_q} o \, \psi \right), \ \tilde{f} = |\det D\psi| \, f \, o \, \psi.$$

The matrix $\tilde{a}_{ij}$ satisfies the uniform ellipticity condition if a_{pq} does. To see this, we define $\zeta_p = (D\varphi^t)\xi$,

or

$$\zeta_p = \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_p} \xi_i$$

Then, since $D\varphi$ and $D\psi = D\varphi^{-1}$ are invertible and bounded away from zero, we have for some constant C >0 that

$$\sum_{i,j=1}^{n} \tilde{a}_{ij} \xi_i \xi_j = |\det D\psi| \sum_{p,q=1}^{n} a_{pq} \zeta_p \zeta_q \geq |\det D\psi| \theta |\zeta|^2 \geq C\theta |\xi|^2$$

Thus, we obtain a problem of the same form as before after the change of variables. Note that we must require that the boundary is C^2 to ensure that $\tilde{a}_{ij}$ is C^1.

It is important to recognize that in changing variables for weak solutions, we need to verify the change of variables for the weak formulation directly and not just for the original PDE. A transformation that is valid for smooth solutions of a PDE is not always valid for weak solutions, which may lack sufficient smoothness to justify the transformation.

We now state a boundary regularity theorem. Unlike the interior regularity theorem, we impose a boundary condition $u \in H_0^1(\Omega)$ on the solution, and we require that the boundary of the domain is smooth. A solution of an elliptic PDE with smooth coefficients and smooth right-hand side is smooth in the interior of its domain of definition, whatever

Notes

its behavior near the boundary; but we cannot expect to obtain smoothness up to the boundary without imposing a smooth boundary condition on the solution and requiring tha0t the boundary is smooth.

THEOREM 9.11. Suppose that Ω is a bounded open set in R^n with C^2 boundary. Assume that $a_{ij} \in C^1(\overline{\Omega})$ and $f \in L^2(\Omega)$. If $u \in H_0^1(\Omega)$ is a weak solution of (9.15) – (9.16), then $u \in H^2(\Omega)$, and

$$\| u \|_{H^2(\Omega)} \leq C\left(\| f \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)}\right)$$

where the constant C depends only on n, Ω and a_{ij}.

PROOF. By use of a partition of unity and a flattening of the boundary, it is sufficient to prove the result for an upper half space $\Omega = \{(x_1,...,x_n) : x_n > 0\}$ space and functions

u,f : $\Omega \rightarrow R$ that are compactly supported in $B_1(0) \cap \overline{\Omega}$. Let $\eta \in C_c^\infty(R^n)$ be a cut-off function such that $0 \leq \eta \leq 1$ and $\eta = 1$ on $B_1(0)$. We will estimate the tangential and normal difference quotients of Du separately. First consider a test function that depends on tangential differences,

$$v = -D_k^{-h}\eta^2 D_k^h u \text{ for k=1,2,……,n-1}$$

Since the trace of u is zero on $\partial\Omega$, the trace of v on $\partial\Omega$ is zero and, by Theorem 3.44, $v \in H_0^1(\Omega)$. Thus we may use v in the definition of weak solution to get (9.21). Exactly the same argument as the one in the proof of Theorem 9.8 gives (9.26). It follows from Theorem 4.53 that the weak derivatives $\partial_k \partial_i u$ exist and satisfy

$$(9.28) \quad \| \partial_k Du \|_{L^2(\Omega)} \leq C\left(\| f \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)}\right) \text{for k=1, 2……, n-1}$$

The only derivative that remains is the second-order normal derivative $\partial_n^2 u$, which we can estimate from the equation. Using (9.15)–(9.16), we have for $\phi \in C_c^\infty(\Omega)$ that

$$\int_\Omega a_{nn}\left(\partial_n u\right)\left(\partial_n \phi\right)dx = -\sum\int_\Omega a_{ij}\left(\partial_i u\right)\left(\partial_j \phi\right)dx + \int_\Omega f\phi dx$$

Where $\sum'$ denotes the sum over $1 \le i, j \le n$ with the term $i = j = n$ omitted. Since $a_{ij} \in C^1(\Omega)$ and $\partial_i u$ is weakly differentiable with respect to x_j unless $i = j = n$ we get, using Proposition 3.21, that

$\int_\Omega a_{nn}\left(\partial_n u\right)\left(\partial_n \phi\right)dx = \Sigma'\int_\Omega\left\{\partial_j\left[a_{ij}\left(\partial_i u\right)\right] + f\right\}\phi\, dx$ for every $\phi \in C_c^\infty(\Omega)$.

It follows that $a_{nn}\left(\partial_n u\right)$ is weakly differentiable with respect to x_n, and

From the uniform ellipticity condition (8.18) with $\xi = e_n$, we have $a_{nn} \ge \theta$. Hence by Proposition 3.21,

$$\partial_n u = \frac{1}{a_{nn}} a_{nn}\partial_n u$$

Is weakly differentiable with respect to x_n with derivative

$$\partial_{nn}^2 u = \frac{1}{a_{nn}}\partial_n\left[a_{nn}\partial_n u\right] + \partial_n\left(\frac{1}{a_{nn}}\right)a_{nn}\partial_n u \in L^2(\Omega)$$

Furthermore, using (9.28) we get an estimate of the same form for $\| \partial_{nn}^2 u \|_{L^2(\Omega)}^2$, so that

$$\| D^2 u \|_{L^2(\Omega)} \le C\left(\| f \|_{L^2(\Omega)}^2 + \| u \|_{L^2(\Omega)}^2\right)$$

The repeated application of these estimates leads to higher-order regularity.

THEOREM 9.12. Suppose that Ω is a bounded open set in R^n with C^{k+2} – boundary. Assume that $a_{ij} \in C^{k+1}\left(\overline{\Omega}\right)$ and $f \in H^k(\Omega)$. If $u \in H_0^1(\Omega)$ is a weak solution of (9.4) –(9.16), then $u \in H^{k+2}(\Omega)$ and

$$\| u \|_{H^{k+2}(\Omega)} \le C\left(\| f \|_{H^k(\Omega)} + \| u \|_{L^2(\Omega)}\right)$$

Where the constant C depends only on n, k, Ω and a_{ij}.

Sobolev embedding then yields the following result.

COROLLARY 9.13, Suppose that that Ω is a bounded open set in R^n with C^∞ boundary. If $a_{ij}, f \in C^\infty(\overline{\Omega})$ and $u \in H_0^1(\Omega)$ is a weak solution of (9.15) – (9.16), then $u \in C^\infty(\overline{\Omega})$

Check your progress

2. Explain about boundary regularity

9.8 SOME FURTHER PERSPECTIVES

This book is to a large extent self-contained, with the restriction that the linear theory — Schauder estimates and Campanato theory — is not presented. The reader is expected to be familiar with functional-analytic tools, like the theory of monotone operators.[4]

The above results give an existence and L^2-regularity theory for second-order, uniformly elliptic PDEs in divergence form. This theory is based on the simple a priori energy estimate for $\|Du\|_{L^2}$ that we obtain by multiplying the equation Lu = f by u, or some derivative of u, and integrating the result by parts.

This theory is a fundamental one, but there is a bewildering variety of approaches to the existence and regularity of solutions of elliptic PDEs. In an attempt to put the above analysis in a broader context, we briefly list some of these approaches and other important results, without any claim to completeness.

L^p-theory: If 1 < p < ∞, there is a similar regularity result that solutions of Lu = f satisfy $u \in W^{2,p}$ if $f \in L^p$. The derivation is not as simple when $p \neq 2$, however, and requires the use of more sophisticated tools from real analysis (such as the L^p-theory of Caldero´n-Zygmundoperators).

Schauder theory: The Schauder theory provides H¨older-estimates similar to those derived in Section 2.7.2 for Laplace's equation, and a corresponding existence theory of solutions u $\in C^{2,\alpha}$ of $Lu = f$ if $f \in C^{0,\alpha}$ and L has Holder continuous coefficients. General linear elliptic PDEs are treated by regarding them as perturbations of constant coefficient PDEs, an approach that works because there is no 'loss of derivatives' in the estimates of the solution. The Holder estimates were originally obtained by the use of potential theory, but other ways to obtain them are now known; for example, by the use of Campanato spaces, which provide H¨older norms in terms of suitable integral norms that are easier to estimate directly.

Perron's method: Perron (1923) showed that solutions of the Dirichlet problem for Laplace's equation can be obtained as the infimum of superharmonic functions or the supremum of subharmonic functions, together with the use of barrier functions to prove that, under suitable assumptions on the boundary, the solution attains the prescribed boundary values. This method is based on maximum principle estimates.

Boundary integral methods: By the use of Green's functions, one can often reduce a linear elliptic BVP to an integral equation on the boundary, and then use the theory of integral equations to study the existence and regularity of solutions. These methods also provide efficient numerical schemes because of the lower dimensionality of the boundary.

Pseudo-differential operators: The Fourier transform provides an effective method for solving linear PDEs with constant coefficients. The theory of pseudo-differential and Fourier-integral operators is a powerful extension of this method that applies to general linear PDEs with variable coefficients, and elliptic PDEs in particular. It is, however, less well suited to the analysis of nonlinear PDEs (although there are nonlinear generlizations, such as the theory of para-differential operators).

Variational methods: Many elliptic PDEs — especially those in divergence form — arise as Euler-Lagrange equations for variational principles. Direct methods in the calculus of variations provide a powerful and general way to analyze such PDEs, both linear and nonlinear.

Notes

Di Giorgi-Nash-Moser: Di Giorgi (1957), Nash (1958), and Moser (1960) showed that weak solutions of a second order elliptic PDE in divergence form with bounded L^∞ coefficients are Holder continuous $\left(C^{0,\alpha}\right)$. This was the key step in developing a regularity theory for minimizers of nonlinear variational principles with elliptic Euler-Lagrange equations. Moser

also obtained a Harnack inequality for weak solutions which is a crucial ingredient of the regularity theory.

Fully nonlinear equations: Krylov and Safonov (1979) obtained a Harnack inequality for second order elliptic equations in nondivergence form. This allowed the development of a regularity theory for fully nonlinear elliptic equations (e.g. second-order equations for u that depend nonlinearly on D^2u). Crandall and Lions (1983) introduced the notion of viscosity

solutions which — despite the name — uses the maximum principle and is based on a comparison with appropriate sub and super solutions This theory applies to fully nonlinear elliptic PDEs, although it is mainly restricted to scalar equations.

Degree theory: Topological methods based on the Leray-Schauder degree of a mapping on a Banach space can be used to prove existence of solutions of various nonlinear elliptic problems . These methods can provide global existence results for large solutions, but often do not give much detailed analytical information about the solutions.

Heat flow methods: Parabolic PDEs, such as $u_t = Lu = f$, are closely connected with the associated elliptic PDEs for stationary solutions, such as $Lu = f$. One may use this connection to obtain solutions of an elliptic PDE as the limit as $t \to \infty$ of solutions of the associated parabolic PDE. For example, Hamilton (1981) introduced the Ricci flow on a manifold, in which the metric approaches a Ricci-flat metric as $t \to \infty$, as a means to understand the topological classification of smooth manifolds, and Perelman (2003) used this approach to prove the Poincare conjecture (that every simply connected, three-dimensional, compact manifold without boundary is homeomorphic to a three-dimensional sphere) and, more generally, the geometrization conjecture of Thurston.

9.9 LET US SUM UP

In this unit we have discussed about The Lax-Milgran theorem, Compactness of the resolvent, The Fredholm alternative, The spectrum of a self-adjoint elliptic operator, Interior regularity, Boundary regularity, The only weak solution of the equation $L*v-\lambda v=0$ is v=0. For every $f\in L^2(\Omega)$ there is a unique weak solution $u\in H_0^1(\Omega)$ of the equation $Lu-\lambda u=f$. In particular, the only solution of $Lu-\lambda u=0$ is u=0. By the use of Green's functions, one can often reduce a linear elliptic BVP to an integral equation on the boundary. The Fourier transform provides an effective method for solving linear PDEs with constant coefficients. Many elliptic PDEs — especially those in divergence form — arise as Euler-Lagrange equations for variational principles. Direct methods in the calculus of variations provide a powerful and general way to analyze such PDEs, both linear and nonlinear.

9.10 KEY WORDS

1.Let H be Hilbert space with inner-product $(.,.):\mathrm{H}\times\mathrm{H}\to\mathrm{R}$, and let $\mathrm{a}:\mathrm{H}\times\mathrm{H}\to\mathrm{R}$, be a bilinear form on H. Assume that there exist constants $\mathrm{C}_1,\mathrm{C}_2>0$ such that

$$C_1\|u\|^2\le a(u,u),|a(u,v)| \text{ for all } u,v\in H$$

2.The Lax-Milgram theorem, for every $f\in H^{-1}(\Omega)$ there is a unique $u\in H_0^1(\Omega)$ such that $\langle f,\phi\rangle=a_\mu(u,\phi)$ for all $v\in H_0^1(\Omega)$

3.If $K\in B\left(L^2(\Omega)\right)$ is defined by (9.7), then the adjoint of K is K^* defined by (9.8). If Ω is a bounded open set, then K is a compact operator.

4.L is a symmetric, uniformly elliptic operator of the form

$$Lu=-\sum_{i,j=1}^{n}\partial_i\left(a_{ij}\partial_j u\right)+cu$$

5.For boundary value problems, it is convenient to consider the regularity of the solution in the interior of the domain and neat the boundary separately.

6.To study the regularity of solutions near the boundary, we localize the problem to a neighborhood of a boundary point by use of a partition of unity.

9.11 QUESTIONS FOR REVIEW

1. Discuss about The Lax-Milgram theorem
2. Discuss about the spectrum of a self adjoint elliptic operator
3. Discuss about Boundary regularity

9.12 SUGGESTED READINGS AND REFERENCES

1.S. L. Ross, Differential Equations, 3rd Edn., Wiley India, 1984.

2.DiBenedetto, Partial Differential Equations, Birkhaüser, 1995.

*3.*L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.

*4.*I.N. Sneddon Elements of Partial Differential Equations McGrawHill 1986.

*5.*R. Churchil & J. Brown, Fourier Series & Boundary Value Problems.

*6.*R.C. McOwen , Partial Differential Equations (Pearson Edu.) 2003.

7.Duchateau and D.W. Zachmann, “Partial Differential Equations,” Schaum, Outline Series, McGraw hill Series.

8.Partial Differential Equations, -Walter A.Strauss

9.Partial Differential Equations,-John K.Hunter

10. Partial Differential Equations,*Erich Mieremann*

11. Partial Differential Equations,-*Victor Ivrii*

9.13 ANSWERS TO CHECK YOUR PROGRESS

1. See section 9.2
2. See section 9.4
3. See section 9.7

UNIT-10 THE HEAT AND SCHRODINGER EQUATIONS PART-1

STRUTURE

10.0 OBJECTIVE

In this unit we will learn and understand about Heat equation, Initial value problem of equation, Schwartz solutions, irreversibility, Non unique ness, Generalized solutions, The Schrodinger equations, Semi groups and groups, Non-autonomus equations.

10.1 INTRODUCTION

Study the solutions of the heat equation satisfy Laplace's equation, Initial value problem of heat equation, and the solution is a spherically symmetric Gaussian with spatial integral equal to one which spreads out and decays as t increases; its width is of the order $\sqrt{t}$ and its height is of the order $t^{-1/2}$. Green's function, Schwartz solutions.

10.2 HEAT EQUATION

The heat, or diffusion, equation is $u_t = \Delta u.$ (10.1)

Section 9.A derives (10.1) as a model of heat flow.

Steady solutions of the heat equation satisfy Laplace's equation.

We have for smooth functions that

$$\Delta u(x) = \lim_{r\to 0^+} f_{B_{r(x)}} \Delta u dx$$

$$= \lim_{r\to 0^+} \frac{n}{r}\frac{\partial}{\partial r}\left[f_{\partial B_{r(x)}} u dS \right]$$

$$= \lim_{r\to 0^+} \frac{2n}{r^2}\left[f_{\partial B_{r(x)}} u dS - u(x) \right]$$

Thus, if u is a solution of the heat equation, then the rate of change of $u(x,t)$ with respect to t at a point x is proportional to the difference between the value of u at x and the average of u over nearby spheres centered at x. The solution decreases in time if its value at a point is greater than the nearby mean and increases if its value is less than the nearby averages. The heat equation therefore describes the evolution of a function towards its mean. As $t \to \infty$ solutions of the heat equation typically approach functions with the mean value property, which are solutions of Laplace's equation.

We will also consider the Schrodinger equation

$$iu_t = -\Delta u$$

This PDE is a dispersive wave equation, which describes a complex wave-field that oscillates with a frequency proportional to the difference between the value of the function and its nearby means.

Check your progress

1. Explain about heat equation.

10.3 THE INITIAL VALUE PROBLEM FOR THE HEAT EQUATION

Consider the initial value problem for $u(x,t)$ where $x \in R^n$

10.2. $y_t = \Delta u$ for $x \in R^n$ and $t > 0$

$$u(x,0) = f(x) \qquad \text{for } x \in R^n$$

We will solve (10.2) explicitly for smooth initial data by use of the Fourier transform, following the presentation in [34]. Some of the main qualitative features illustrated by this solution are the smoothing effect of the heat equation, the irreversibility of its semiflow, and the need to impose a growth condition as $|x| \to \infty$ in order to pick out a unique solution.

10.4 SCHWARTZ SOLUTIONS:

Assume first that the initial data $f : R^n \to R$ is a smooth, rapidly decreasing, real-valued Schwartz function $f \in S$. The solution we construct is also a Schwartz function of x at later times $t > 0$, and we will regard it as a function of time with values in S. This is analogous to the geometrical interpretation of a first-order system of ODEs, in which the finite- dimensional phase space of the ODE is replaced by the infinite-dimensional function space S; we then think of a solution of the heat equation as a parametrized curve in the vector space S. A similar viewpoint is useful for many evolutionary PDEs, where the Schwartz space may be replaced other function spaces (for example, Sobolev spaces).

By a convenient abuse of notation, we use the same symbol u to denote the scalar-valued function $u(x,t)$, where $u : R^n \times [0,\infty) \to R,$ and the associated vector- valued function $u(t)$, where $u : [0,\infty) \to S.$ We write the vector-valued function corresponding to the associated scalar-valued function as

Notes

$u(t) = u(.,t)$

DEFINITION 10.1. Suppose that (a,b) is an open interval in R. A function $u:(a,b) \to S$ is continuous at $t \in (a,b)$ if

$$u(t+h) \to u(t) \text{ in } S \text{ as } h \to 0,$$

And differentiable at $t \in (a,b)$ if there exists a function $v \in R$ such that

$$\frac{u(t+h)-u(t)}{h} \to v \text{ in } S \text{ as } h \to 0.$$

The derivative v of u at t is denoted by $u_t(t)$, and if u is differentiable for every $t \in (a,b)$, then $u_t:(a,b) \to S$ denotes the map $u_t : t \to u_t(t)$

In other words, u is continuous at t if

$$u(t) = S - \lim_{h \to 0} u(t+h)$$

and u is differentiable at t with derivative $u_t(t)$ if

$$u(t) = S - \lim_{h \to 0} \frac{u(t+h)-u(t)}{h}$$

We will refer to this derivative as a strong derivative if it is understood that we are considering -valued functions and we want to emphasize that the derivative is defined as the limit of difference quotients In S.

We define spaces of differentiable Schwartz-valued functions in the natural way. For half-open or closed intervals, we make the obvious modifications to left or right limits at an endpoint.

DEFINITION 10.2. The space $C([a,b];S)$ consists of the continuous functions

$$u:[a,b] \to S.$$

The space $C^k(a,b;S)$ consists of functions $u:(a,b) \to S$ that are k-times strongly differentiable in (a,b) with continuous strong derivatives $\partial_t^j u \in C(a,b;S)$ for $0 \le j \le k$, and $C^\infty(a,b;S)$ is the space of functions with continuous strong derivatives of all orders.

Here we write $C(a,b;S)$ rather than $C((a,b);S)$ when we consider functions defined on the open interval (a,b). The next proposition describes the relationship between the C^1- strong derivative and the pointwise time-derivative.

PROPOSITION 10.3. Suppose That $u \in C(a,b;S)$ where $u(t)=u(.,t)$. Then $u \in C^1(a,b;S)$ if and only if:

1. The pointwise partial derivative $\partial_t u(x,t)$ exists for every $x \in R^n$ and $t \in (a,b;)$

2. $\partial_t u(.,t) \in S$ for every $t \in (a,b)$;

3. The map $t \mapsto \partial_t u(\,.,t)$ belongs $C(a,b;S)$.

PROOF: The convergence of functions in S implies uniform pointwise convergence. Thus, if $u(t)=u(.,t)$ is strongly continuously differentiable, then the point- wise partial derivative $\partial_t u(x,t)$ exists for every $x \in R^n$ and $\partial_t u(.,t)=u_t(t) \in S$, so $\partial_t u \in C(a,b;S)$

Conversely, if a pointwise partial derivative with the given properties exist, then for each $x \in R^n$

$$\frac{u(x,t+h)-u(x,t)}{h}-\partial_1 u(x,t)=\frac{1}{h}\int_t^{t+h}\left[\partial_s u(x,s)-\partial_t u(x,t)\right] \text{ ds.}$$

Since the integrand is a smooth rapidly decreasing function, it follows from the dominated convergence Theorem that we may differentiate under the integral sign with respect to x, to get

$$x^\alpha \partial^\beta\left[\frac{u(x,t+h)-u(x,t)}{h}\right]=\frac{1}{h}\int_t^{t+h} x^\alpha \partial^\beta\left[\partial_s u(x,s)-\partial_t u(x,t)\right]\text{ds,}$$

Hence, if $\| \,.\, \| \alpha,\beta$ is a schwartz seminorm , we have

$$\left\|\frac{u(t+h)-u(t)}{h}-\partial_t u(.,t)\right\|_{\alpha,\beta} \le \frac{1}{|h|}\left|\int_t^{t+h}\left\|\partial_s u(.,s)-\partial_t u(.,t)\right\|_{\alpha,\beta} ds\right|$$

$$\leq \max_{t \leq 8 \leq t+h} \left\| \partial_s u(.,s) - \partial_t u(.,t) \right\|_{\alpha,\beta},$$

And since $\partial_t u \in C(a,b;S)$

$$\lim_{h \to 0} \left\| \frac{u(t+h)-u(t)}{h} - \partial_t u(.,t) \right\|_{\alpha,\beta} = 0.$$

If follows that

$$S - \lim_{h \to 0} \left[\frac{u(t+h)-u(t)}{h} \right] = \partial_t u(.,t),$$

So u is strongly differentiable and $u_t = \partial_t u \in C(a,;S)$.

We interpret the initial value problem (10.2) for the heat equation as follows: A solution is a function $u:[0,\infty) \to S$ that is continuous for $t \geq 0$, so that it makes sense to impose the initial condition at $t = 0$, and continuously differentiable for $t > 0$, so that it makes sense to impose the PDE point wise in t. That is, for $t > 0$, the strong derivative $u_t(t)$ is required to exist and equal $\Delta u(t)$ where $\Delta : S \to S$ is the Laplactian operator.

THEOREM 10.4. If $f \in S$, there is a unique solution

(10.3) $$u \in C([0,\infty);S) \cap C^1(0,\infty;S)$$

Of (10.2) Furthermore, $u \in C^\infty([0,\infty);S)$. The spatial fourier transform of the solution is given by

(10.4) $$u(k,t) = f(k)e^{t|k|^2},$$

and for $t > 0$ the solution is given by

(10.5) $$u(x,t) = \int_{R^n} \Gamma(x-y,t) f(y) dy$$

Where

(10.6) $$\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2 4t}.$$

PROOF: Since the spatial Fourier transform F is a continuous linear map on S with continuous inverse, the time-derivative of u exists if and only if the time derivative of $u^\wedge = Fu$ exists, and

$$F(u_t) = (Fu)_t.$$

Moreover, $u \in C([0,\infty);S)$ if and only if $u^\wedge \in C([0,\infty);S)$, and

$u \in C^k(0,\infty;S)$ if and only if $u^\wedge \in C^k(0,\infty;S)$.

Taking the Fourier transform of (10.2) with respect to x, we find that

$u(x,t)$ is a solution with the regularity in (10.3) if and only if $u^\wedge(k,t)$

satisfies

(10.7) $u^\wedge{}_t = -|k|^2 u^\wedge, \qquad u^\wedge(0) = f^\wedge,$

$u^\wedge \in C([0,\infty);S) \cap C^1(0,\infty;S).$

Equation (10.7) has the unique solution (10.4)

To show this in detail, suppose first that $u^\wedge$ satisfies (10.7). Then,

from Propo- sition 10.3, the scalar-valued function $u^\wedge(k,t)$ is

pointwise-differentiable with respect to t in t > 0 and continuous in

$t \geq 0$ for each fixed $k \in R^n$.

Solving the ODE (10.7) with k as a parameter, we find that $u^\wedge$

must be given by (10.4).

Conversely, we claim that the function defined by (10.4) is strongly

differentiable with derivative

(10.8) $$u^\wedge{}_t(k,t) = -|k|^2 f^\wedge(k) e^{-t|k|^2}.$$

To prove this claim, note hat if $\alpha, \beta \in N_0^n$ are any multi-indices, the

function

$$k^\alpha \partial^\beta \left[u^\wedge(k,t+h) - u^\wedge(k,t) \right]$$

Has the form

$$a^\wedge(k,t)\left[e^{-h|k|^2} - 1\right]e^{-t|k|^2} + h\sum_{i=0}^{|\beta|-1} h^i b^\wedge{}_i(k,t)e^{-(t+h)|k|^2}$$

Where $a^\wedge(.,t), b^\wedge{}_i(.,t) \in S$, so taking the supremum of this expression

we see that

$$\|u^\wedge(t+h)u^\wedge(t)\|_{\alpha\beta} \to 0 \text{ as } h \to 0.$$

Thus, $u^\wedge(.,t)$ is a continuous S-valued function in $t \geq 0$ for every

$f^\wedge \in S$. By a similar argument, the pointwise partial derivative

$u^\wedge{}_t(.,t)$ in (10.8) is a continuous S-valued function.

Thus, Proposition (10.3) implies that $u^\wedge$ is a strongly continuously differentiable function that satisfies (10.7).

Hence $u = F^{-1}\left[u^\wedge\right]$ satisfies (10.3) and is a solution of (10.2).

Moreover, using induction and proposition (10.3) we see in a similar way that $u \in C^\infty\left([0,\infty);S\right)$.

Finally, from example 10.65 we have

$$F^{-1}\left[e^{-t|k|^2}\right] = \left(\frac{\pi}{t}\right)^{n/2} e^{-|x|^2/4t}.$$

Taking the inverse Fourier transform of (10.4) and using the convolution Theorem, Theorem 10.67, we get (10.5)–(10.6).

The function $\Gamma(x,t)$ in (10.6) is called the Green's function or fundamental solution of the heat equation in R^n. It is a C^∞ – function of (x,t) in $R^n \times (0,\infty)$, and one can verify by direct computation that

(10.9) $$\Gamma_t = \Delta\Gamma \text{ if } t > 0$$

Also, since $\Gamma(.,t)$ is a family of Gaussian mollifiers, we have

$$\Gamma(.,t) \to \delta \text{ in } S' \text{ as } t \to 0^+.$$

Thus, we can interpret $\Gamma(x,t)$ as the solution of the heat equation due to an initial point source located at $x=0$. The solution is a spherically symmetric Gaussian with spatial integral equal to one which spreads out and decays as t increases; its width is of the order $\sqrt{t}$ and its height is of the order $t^{-1/2}$.

The solution at time *t* is given by convolution of the initial data with $\Gamma(.,t)$. For any $f \in S$, this gives a smooth classical solution $u \in C^\infty\left(R^n \times [0,\infty)\right)$ of the heat equation which satisfies it pointwise in $t \geq 0$.

10.1.2. Smoothing. Equation (10.5) also gives solutions of (10.2) for initial data that is not smooth. To be specific, we suppose that $f \in L^p$, although one can also consider more general data that does not grow too rapidly at infinity.

Theorem10.5. *Suppose that* $1 \le p \le \infty$ *and* $f \in L^p(R^n)$. *Define*

$$u : R^n \times (0, \infty) \to R$$

By (10.5) where Γ is given in (10.6) then $u \in C_0^\infty(R^n \times (0,\infty))$ and $u_t = \Delta u$ in $t > 0$. If $1 \le p < \infty$, then $u(.,t) \to f$ in L^p as $t \to 0^+$.

PROOF. The Green's function Γ in (10.6) satisfies (10.9), and $\Gamma(.,t) \in L^q$ for every $1 \le q \le \infty$,, together with all of its derivatives. The dominated convergence Theorem and Holder's inequality imply that if $f \in L^p$ and $t > 0$, we can differentiate under the integral sign in (10.10) arbitrarily often with respect to (x,t) and that all of these derivatives approach zero as $|x| \to \infty$. Thus, u is a smooth, decaying solution of the heat equation in $t > 0$. Moreover, $\Gamma^t(x) = \Gamma(x,t)$ is a family of Gaussian mollifiers and therefore for $1 \le p < \infty$ we have from Theorem 1.28 that $u(.,t) = \Gamma^t * f \to f$ in L^p as $t \to 0^+$.

The heat equation therefore immediately smooths any initial data $f \in L^p(R^n)$ to a function $u(.,t) \in C_0^\infty(R^n)$. From the Fourier perspective, the smoothing is a consequence of the very rapid damping of the high-wavenumber modes at a rate proportional to $e^{-t|k|^2}$ for wave numbers $|k|$, which physically is caused by the diffusion of thermal energy from hot to cold parts of spatial oscillations.

Once the solution becomes smooth in space it also becomes smooth in time. In general, however, the solution is not (right) differentiable with respect to t at $t = 0$, and for rough initial data it satisfies the initial condition in an L^p-sense, but not necessarily pointwise.

Check your progress

2.Explain about Schwartz solutions

Notes

----

3. Prove proposition 10.3

----

4. Prove theorem 10.4

10.5 IRREVERSIBILITY

For general 'final' data $f \in S$, we cannot solve the heat equation backward in time to obtain a solution $u:[-T,0] \to S$, however small we choose $T > 0$. The same argument as the one in the proof of Theorem 10.4 implies that any such solution would be given by (10.4). If, for example, we take $f \in S$ such that

$$f^\wedge(k) = e^{-\sqrt{1+|k|^2}}$$

then the corresponding solutions

$$u(k,t) = e^{-t|k|^2 - \sqrt{1+|k|^2}}$$

grows exponentially as $|k| \to \infty$ for every $t < 0$, and therefore $u(t)$ does not belong to S (or even S'). Physically, this means that the temperature distribution f cannot arise by thermal diffusion from any previous temperature distribution in S (or S'). The heat equation does, however, have a backward uniqueness property, meaning that if f arises from a previous temperature distribution, then (under appropriate assumptions) that distribution is unique [9].

Equivalently, making the time-reversal $t \mapsto -t$, we see that Schwartz-valued solutions of the initial value problem for the backward heat equation

$$u_t = -\Delta u \quad t > 0, \qquad u(x,0) = f(x)$$

do not exist for every $f \in S$. Moreover, there is a loss of continuous dependence of the solution on the data.

Example 10.6. Consider the one-dimensional heat equation $u_t = u_{xx}$ with initial data

$$f_n(x) = e^{-n} \sin(nx)$$

and corresponding solution

$$u_n(x,t) = e^{-n} \sin(nx) e^{n^2 t}.$$

Then $f_n \to 0$ uniformly together with of all its spatial derivatives as $n \to \infty$, but sup

$$\sup_{x \in R} |u_n(x,t)| \to \infty$$

as $n \to \infty$ for any $t > 0$. Thus, the solution does not depend continuously on the initial data in $C_b^\infty(R^n)$. Multiplying the initial data f_n by e^{-x^2}, we can get an example of the loss of continuous dependence in S.

It is possible to obtain a well-posed initial value problem for the backward heat equation by restricting the initial data to a small enough space with a strong enough norm — for example, to a suitable Gevrey space of C^∞ – functions whose spatial derivatives decay at a sufficiently fast rate as their order tends to infinity. These restrictions, however, limit the size of derivatives of all orders, and they are too severe to be useful in applications.

Nevertheless, the backward heat equation is of interest as an inverse problem, namely: Find the temperature distribution at a previous time that gives rise to an observed temperature distribution at the present time. There is a loss of continuous dependence in any reasonable function space for applications, because thermal diffusion damps out large, rapid variations in a

Notes

previous temperature distribution leading to an imperceptible effect on an observed distribution. Special methods — such as Tychonoff regularization — must be used to formulate such ill-posed inverse problems and develop numerical schemes to solve them.

10.6 NON UNIQUENESS

A solution $u(x,t)$ of the initial value problem for the heat equation on R^n is not unique without the imposition of a suitable growth condition as $|x|\to\infty$. In the above analysis, this was provided by the requirement that $u(.,)\in S$, but the much weaker condition that u grows more slowly than $Ce^{a|x|^2}$ as $|x|\to\infty$ for some constants C, a is sufficient to imply uniqueness [9].

EXAMPLE 10.7. Consider, for simplicity, the one-dimensional heat equation

$$u_t = u_{xx}.$$

As observed by Tychonoff , a formal power series expansion with respect to x gives the solution

$$u(x,t)\sum_{n=0}^{\infty}\frac{g^{(n)}(t)x^{2n}}{(2n)!}$$

for some function $g\in C^{\infty}(R^{+}.)$ We can construct a nonzero solution with zero initial data by choosing $g(t)$ to be a nonzero C^{∞} -function all of whose derivatives vanish at $t=0$ in such a way that this series converges uniformly for x in compact subsets of R and $t>0$ to a solution of the heat equation. This is the case, for example, if

$$g(t)=\exp\left(\frac{1}{t^2}\right).$$

The resulting solution, however, grows very rapidly as $|x|\to\infty$.

A physical interpretation of this nonuniqueness it is that heat can diffuse from infinity into an unbounded region of initially zero temperature if the solution grows sufficiently quickly. Mathematically, the nonuniqueness is a consequence of the the fact that the initial condition is imposed on a characteristic surface $t = 0$ of the heat equation, meaning that the heat equation does not determine the second- order normal (time) derivative u_{tt} on $t = 0$ in terms of the second-order tangential (spatial) derivatives u, Du, D^2u.

According to the Cauchy-Kowalewski Theorem [14], any non-characteristic Cauchy problem with analytic initial data has a unique local analytic solution. If $t \in R$ denotes the normal variable and $x \in R^n$ the transverse variable, then in solving the PDE by a power series expansion in t we exchange one t-derivative for one x-derivative and the convergence of the Taylor series in x for the analytic initial data implies the convergence of the series for the solution in t. This existence and uniqueness fails for a characteristic initial value problem, such as the one for the heat equation.

The Cauchy-Kowalewski Theorem is not as useful as its apparent generality sug- gests because it does not imply anything about the stability or existence of solutions under non-analytic perturbations, even arbitrarily smooth ones. For example, the Cauchy-Kowalewski Theorem is equally applicable to the initial value problem for the wave equation

$$u_{tt} = u_{xx}, u(x,0) = f(x),$$

which is well-posed in every Sobolev space $H^s(R)$, and the initial value problem for the Laplace equation

$$u_{tt} = u_{xx}, u(x,0) = f(x),$$

which is ill-posed in every Sobolev space $H^s(R)$.

10.7 GENERALIZED SOLUTIONS

In this section we obtain generalized solutions of the initial value problem of the heat equation as a limit of the smooth solutions

Notes

constructed above. In order to do this, we require estimates on the smooth solutions which ensure that the convergence of initial data in suitable norms implies the convergence of the corresponding solution.

10.7.1. Estimates for the Heat equation. Solutions of the heat equation satisfy two basic spatial estimates, one in L^2 and the L^∞. The L^2 estimate follows from the Fourier representation, and the L^1 estimate follows from the spatial representation. For $1 \le p < \infty$, we let

$$\| f \|_{lp} = \left(\int_{R^n} | f |^p \, dx \right)^{1/p}$$

denote the spatial L^p-norm of a function f; also $\| f \|_{L^\infty}$ denotes the maximum or essential supremum of $|f|$.

Theorem 10.8. *Let $u : [0,\infty) \to S(R^n)$ be the solution of (10.2) constructed in Theorem 10.4 and $t > 0$. Then*

$$\|u(t)\|_{L^2} \le \| f \|_{L^2}, \|u(t)\|_{L^\infty} \le \frac{1}{(4\pi t)^{n/2}} \| f \|_{L^1}.$$

Proof:.By Parseval's inequality and (10.4),

$$\|u(t)\|_{L^2} = (2\pi)^n \|u(t)\|_{L^2} = (2\pi)^n \left\| e^{-t|k|^2} f^\wedge \right\|_{L^2} \le (2\pi)^n \| f^\wedge \|_{L^2 =} \| f \|_{L^2},$$

Which gives the first inequality. From (10.5),

$$|u(x,t)| \le \left(\sup_{x \in \mathbb{R}^n} |\Gamma(x,t)| \right) \int_{\mathbb{R}^n} | f(y)| dy,$$

and from (5.6)

$$|\Gamma(x,t)| = \frac{1}{(4\pi t)^{n/2}}.$$

The second inequality then follows.

Using the Riesz-Thorin Theorem, Theorem 10.72, it follows by interpolation be- tween $(p, p') = (2,2)$ and $(p, p') = (\infty,1)$ that for $2 \le p \le \infty$

(10.10) $$\|u(t)\|_{L^p} \le \frac{1}{(4\pi t)^{n(1/2-1/p)}} \| f \|_{L^{p'}}.$$

This estimate is not particularly useful for the heat equation, because we can de- rive stronger parabolic estimates for $\|Du\|_{L^2}$, but the analogous estimate for the Schrodinger equation is very useful.

A generalization of the L^2-estimate holds in any Sobolev space H^8 of functions with s spatial L^2-derivatives (see Section 10.C for their definition). Such estimates of L^2-norms of solutions or their derivative are typically referred to as energy es- timates, although the corresponding L^2-norms may not correspond to a physical

Finally, here is the question to the answer posed above: Do you spell your name with a "V". Herr Wagner?

energy. In the case of the heat equation, the thermal energy (measured from a zero-point energy at $u=0$) is proportional to the integral of u.

THEOREM 10.9. Suppose that $f \in S$ and $u \in C^{\infty}([0,\infty);S)$ is the solution of (10.2). then for any $s \in R$ and $t \geq 0$

$$\|u(t)\|_{H^s} \leq \|f\|_{H^s}.$$

Proof. Using (10.4) and Parseval's identity, and writing $\langle k\rangle = \left(1+|k|^2\right)^{1/2}$, we find that

$$\|u(t)\|_{H^s} = (2\pi)^n \left\|\langle k\rangle^s e^{-t|k|^2} f^{\wedge}\right\|_{L^2} \leq (2\pi)^n \left\|\langle k\rangle^s f^{\wedge}\right\|_{L^2} \|f\| H^s.$$

We can also derive this H^s-estimate, together with an additional a space-time estimate for Du, directly from the equation without using the explicit solution. We will use this estimate later to construct solutions of a general parabolic PDE by the Galerkin method, so we derive it here directly.

For $1 \leq p \leq \infty$ and $T>0$, the L^p-in-time-H^s-in-space norm of a function $u \in C([0,T];S)$ is given by

$$\|u\|_{L^p([0,T];H^s)} = \left(\int_0^T \|u(t)\|_{H^s}^p \, dt\right)^{1/p}.$$

The maximum-in-time-H^s-in-space norm of u is

(10.11) $$\|u\|_{C([0,T];H^s)} = \max_{t\in[0,T]} \|u(t)\|_{H^s}.$$

In particular, if $\Lambda = (I-\Delta)^{1/2}$ is the spatial operator defined in (10.75), then

$$\|u\|_{L^2([0,T];H^s)} = \left(\int_0^T \int_{\mathbb{R}^n} \left|\Lambda^s u(x,t)\right|^2 dxdt \right)^{1/2}.$$

THEOREM 10.10. Suppose that $f \in S$ and $u \in C^\infty([0,T];S)$ is the solution of (10.2.). Then for any $s \in R$

$$\|u\|_{C([0.T];H^s)} \le \|f\|_{H^s}, \qquad \|Du\|_{L^2([0.T];H^s)} \le \frac{1}{\sqrt{2}}\|f\|_{H^s}.$$

PROOF. Let $v = \Lambda^s u$. Then, since $\Lambda^s : S \to S$ is continuous and commutes with Δ,

$$v_t = \Delta v, \qquad v(0) = g$$

where $g = \Lambda^s f$. Multiplying this equation by v, integrating the result over R^n, and using the divergence Theorem (justified by the continuous differentiability in time and the smoothness and decay in space of v), we get

$$\frac{1}{2}\frac{d}{dt}\int v^2 dx = -\int |Dv|^2 dx.$$

Integrating this equation with respect to t, we obtain for nay $T > 0$ that

(10.12) $$\frac{1}{2}\int v^2(T)\,dx + \int_0^T |Dv(t)|^2 dxdt = \frac{1}{2}\int g^2 dx.$$

Thus,

$$\max_{t\in[0,T]} \int v^2(t)\,dx \le \int g^2 dx, \qquad \int_0^T \int |Dv(t)|^2\, dxdt \le \frac{1}{2}\int g^2 dx,$$

And the result follows.

10.2.2 H^s-solutions. In this section we use the above estimates to obtain generalized solutions of the heat equation as a limit of smooth solutions (10.5). In defining generalized solutions, it is convenient to restrict attention to a finite, but arbitrary, time-interval $[0,T]$ where $T > 0$. For $s \in R$, let $C([0,T];H^s)$ denote the Banach space of continuous H^s-valued functions $u:[0,T] \to H^S$ equipped with the norm (10.11).

Definition 10.11. Suppose that $T > 0, s \in R$ and $f \in H^s$. A function

$$u \in C([0,T]; H^s)$$

is a generalized solution of (10.2) if there exists a sequence of Schwartz-solutions $u_n : [0,T] \to S$ such that $u_n \to u$ in $C([0,T]; H^s)$ as $n \to \infty$

According to the next Theorem, there is a unique generalized solution defined on any time interval $[0,T]$ and therefore on $[0,\infty)$.

Theorem 10.12. *Suppose that* $T > 0, s \in R$ and $f \in H^s(R^n)$..

Then there is a unique generalized solution $u \in C([0,T]; H^s)$ *of (10.2). The solution is given by (10.4).*

Proof. Since S is dense in H^s, there is a sequence of functions $f_n \in S$ such that $f_n \to f$ in H^s. Let $u_n \in C([0,T]; S)$ be the solution of (10.2) with initial data f_n. Then, by linearity, $u_n - u_m$ is the solution with initial data $f_n - f_m$, and Theorem 10.9 implies that

$$\sup_{t \in [0,T]} \| u_n(t) - u_m(t) \|_{H^s} \le \| f_n - f_m \|_{H^s}.$$

Hence, $\{u_n\}$ is a Cauchy sequence in $C([0,T]; H^s)$ and therefore there exists a generalized solution $u \in C([0,T]; H^s)$ such that $u_n \to u$ as $n \to \infty$.

Suppose that $f, g \in H^s$ and $u, v, \in C([0,T]; H^s)$ are generalized solutions with $u(0) = f, v(0) = g$. if $u_n, v_n \in C([0,T]; S)$ are approximate solutions with $u_n(0) = f_n, u_n(0) = g_n$, then

$$\begin{aligned} \| u(t) - v(t) \|_{H^s} &\le \| u(t) - u_n(t) \|_{H^s} + \| u_n(t) - v_n(t) \|_{H^s} + \| v_n(t) - v(t) \|_{H^s} \\ &\le \| u(t) - u_n(t) \|_{H^s} + \| f_n - g_n \|_{H^s} + \| v_n(t) - v(t) \|_{H^s} \end{aligned}$$

Taking the limit of this inequality as $n \to \infty$, we find that

$$\| u(t) - v(t) \|_{H^s} \le \| f - g \|_{H^s}.$$

In particular, if $f = g$ then $u = v$, so a generalized solution is unique.

Notes

Finally, from (10.4) we have

$$u_n(k,t) = e^{-t|k|^2} f^\wedge{}_n(k).$$

Taking the limit of this expression in $C\left([0,T];H^{\wedge s}\right)$ as $n \to \infty$,

where $H^{\wedge s}$ is the weighted L^2 - space (10.74), we get the same

expression for u.

We may obtain additional regularity of generalized solutions in time by use of the equation; roughly speaking, we can trade two space-derivatives for one time- derivative.

Proposition 10.13. *Suppose that* $T > 0, s \in R$ *and* $f \in H^s(R^n)$.

If $u \in C\left([0,T];H^s\right)$ *is a generalized solution of (10.2), then*

$u \in C^1\left([0,T];H^{s-2}\right)$ *and*

$$u_t = \Delta u \text{ in } C\left([0,T];H^{s-2}\right)$$

Proof. Since u is a generalized solution, there is a sequence of smooth so- lutions $u_n \in C^\infty\left([0,T];S\right)$ such that $u_n \to u$ in $C\left([0,T];H^s\right)$ as $n \to \infty$. These solutions satisfy $u_{nt} = \Delta u_n$. Since $\Delta : H^s \to H^{s-2}$ is bounded and $\{u_n\}$ is Cauchy in H^s, we see that $\{u_{nt}\}$ is Cauchy in $C\left([0,T];H^{s-2}\right)$. Hence there exists $v \in C\left([0,T];H^{s-2}\right)$ such that $u_{nt} \to v$ in $C\left([0,T];H^{s-2}\right)$. We claim that $v = u_t$. For each $n \in \mathbb{N}$ and $h \neq 0$ we have

$$\frac{u_n(t+h) - u_n(t)}{h} = \frac{1}{h}\int_t^{t+h} u_{ns}(s)ds \text{ in } C\left([0,T];S\right),$$

And in the limit $n \to \infty$, we get that

$$\frac{u_n(t+h) - u(t)}{h} = \frac{1}{h}\int_t^{t+h} v(s)ds \text{ in } C\left([0,T];H^{s-2}\right).$$

Taking the limit as $h \to 0$ of this equation we find that $u_t = v$ and

$$u \in C\left([0,T];H^s\right) \cap C^1\left([0,T];H^{s-2}\right).$$

Moreover, taking the limit of $u_{nt} = \Delta u_n$ we get $u_t = \Delta u$ in

$C([0,T];H^{s-2})$.

More generally, a similar argument shows that $u \in C^k([0,T];H^{s-2k})$. for any $k \in$. In contrast with the case of ODEs, the time derivative of the solution lies in a different space than the solution itself: u takes values in H^s, but u_t takes values in H^{s-2}. This feature is typical for PDEs when — as is usually the case — one considers solutions that take values in Banach spaces whose norms depend on only finitely many derivatives. It did not arise for Schwartz-valued solutions, since differentiation is a continuous operation on S.

The above proposition did not use any special properties of the heat equation. For $t > 0$, solutions have greatly improved regularity as a result of the smoothing effect of the evolution

Proposition 10.14. If $u \in C([0,T];H^s)$ *is a generalized solution of (10.2), where* $f \in H^s$ *for some* $s \in R$, *then* $u \in C^\infty((0,T];H^\infty)$ *where* H^∞ *is defined in (10.76).*

Proof. $s \in R, f \in H^s$, and $t > 0$, then (10.4) implies that $u(t) \in H^{\wedge r}$ for every $r \in R$, and therefore $u(t) \in H^\infty$. It follows from the equation that $u \in C^\infty(0,\infty;H^\infty)$.

For general H^s-initial data, however, we cannot expect any improved regularity in time at $t = 0$ beyond $u \in C^k([0,T];H^{s-2k})$. The H^∞ spatial regularity stated here is not optimal; for example, one can prove [9] that the solution is a real-analytic function of x for $t > 0$, although it is not necessarily a real-analytic function of t.

Check your progress

1. Prove theorem 10.9

10.8 THE SCHRODINGER EQUATION

The initial value problem for the schrodinger equation is

(10.13) $$iu_t = -\Delta u \text{ for } x \in R^n \text{ and } t \in R,$$

$$u(x,0) = f(x) \text{ for } x \in R^n,$$

where $u : R^n \times R \to C$ is a complex-valued function. A solution of the Schrodinger equation is the amplitude function of a quantum mechanical particle moving freely in R^n. The function $|u(.,t)|^2$ is proportional to the spatial probability density of the particle.

More generally, a particle moving in a potential $V : R^n \to R$ satisfies the schrodinger equation

(10.14) $$iu_t = -\Delta u + V(x)u.$$

Unlike the free Schrödinger equation (10.13), this equation has variable coefficients and it cannot be solved explicitly for general potentials V.

Formally, the Schrödinger equation (10.13) is obtained by the transformation $t \mapsto -it$ of the heat equation to 'imaginary time.' The analytical properties of the heat and Schrodinger equations are, however, completely different and it is interesting to compare them.

The Fourier solution of (10.13) is

(10.15) $$u(k,t) = e^{-it|k|^2 f(k)}.$$

The key difference from the heat equation is that these Fourier modes oscillate instead of decay in time, and higher wavenumber modes oscillate faster in time. As a result, there is no smoothing of the initial data (measuring smoothness in the L^2-scale of Sobolev spaces H^s) and we can solve the Schrodinger equation both forward and backward in time.

Theorem 10.15. For any $f \in S$ there is a unique solution $u \in C^\infty(R;S)$ of (5.13). The spatial Fourier transform of the solution is given by (5.15), and

$$u(x,t)\int \Gamma(x-y,t)\, f(y)\,dy$$

Where

$$\Gamma(x,t) = \frac{1}{(4\pi it)^{n/2}} e^{-i|x|^2/4t}.$$

We get analogous L^p estimates for the schrodinger equation to the ones for the heat equation.

THEOREM 10.16. Suppose that $f \in S$ and $u \in C^\infty(\mathbb{R};S)$ is the solution of (10.13). then for all $t \in \mathbb{R}$,

$$\|u(t)\|_{L^2} \le \|f\|_{L^2}, \|u(t)\|_{L^\infty} \frac{1}{(4\pi|t|)^{n/2}} \|f\|_{L^1},$$

and for $2 < p < \infty$,

(10.16) $$\|u(t)\|_{L^p} \le \frac{1}{(4\pi|t|)^{n(1/2-1/p)}} \|f\|_{L^{p'}}.$$

Solutions of the Schr¨odinger equation do not satisfy a space-time estimate anal- ogous to the parabolic estimate (10.12) in which we 'gain' a spatial derivative. In- stead, we get only that the H^s-norm is conserved. Solutions do satisfy a weaker space-time estimate, called a Strichartz estimate, which we derive in Section 10.6.1. The conservation of the H^s-norm follows from the Fourier representation (10.110), but let us prove it directly from the equation.

THEOREM 10.17. Suppose that $f \in S$ and $u \in C^\infty(R;S)$ is the solution of (10.13). then for any $s \in R$

$$\|u(t)\|_{H^s} = \|f\|_{H^s} \text{ for every } t \in R.$$

PROOF. Let $v = \Lambda^s u,$ so that $\|u(t)\|_{H^s} = \|v(t)\|_{L^2}$. Then

$$iv_t = -\Delta v$$

and $v(0) = \Lambda^s f.$ Multiplying this PDE by the conjugate $\bar{v}$ and subtracting the complex conjugate of the result, we get

$$i\left(\bar{v}v_t + v\bar{v}_t\right) = v\Delta\bar{v} - \bar{v}\Delta v.$$

We may rewrite this equation as

$$\partial_t |v|^2 + \nabla.\left[i\left(vD\bar{v} - \bar{v}Dv\right)\right] = 0.$$

If $v = u,$ this is the equation of conservation of probability where $|u|^2$ is the probability density and $i\left(uD\bar{u} - \bar{u}Du\right)$ is the probability flux.

Notes

Integrating the equation over R^n and using the spatial decay of v, we get

$$\frac{d}{dt}\int |v|^2\, dx = 0,$$

And the result follows.

We say that a function $u \in C\left(R; H^s\right)$ is a generalized solution of (10.13) if it is the limit of smooth schwartz-valued solutions uniformly on compact time intervals. The existence of such solutions follows form the preceding H^s -estimates for smooth solutions.

THEOREM 10.18. Suppose that $s \in R$ and $f \in H^s\left(R^n\right)$. Then there is a unique generalized solution $u \in C\left(R; H^s\right)$ of (10.13) given by

$$u(k) = e^{-it|k|^2} f^\wedge(k).$$

Moreover, for any $k \in N$, we have $u \in C^k\left(R; H^{s-2k}\right)$.

Unlike the heat equation, there is no smoothing of the solution and there is no $H^s -$ regularity for $t \neq 0$ beyond what is stated in this Theorem.

10.9 SEMIGROUPS AND GROUPS

The solution of an $n \times n$ linear first-order system of ODEs for $\vec{u}(t) \in R^n$,

$$\vec{u_t} = A\vec{u},$$

May be written as

$$\vec{u}(t) = e^{tA}\vec{u}(0) \qquad -\infty < t < \infty$$

Where $e^{tA}: R^n \to R^n$ is the matrix exponential of tA. The finite-dimensionality of the phase space R^n is not crucial here. As we discuss next, similar results hold for any linear ODE in a Banach space generated by a bounded linear operator.

10.4.1 Uniformly continuous groups. Suppose that X is a Banch space. We denote by $L(X)$ the Banch space of bounded linear operators $A: X \to X$ equipped with the operator norm

$$\|A\|_{L(x)} = \sup_{u \in X\setminus\{0\}} \frac{\|Au\|x}{\|u\|x}.$$

We say that a sequence of bounded linear operators converges uniformly if it converges with respect to the operator norm.

For $A \in L(X)$ and $t \in R$, we define the operator exponential by the series

(10.17)

$$e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + ... + \frac{1}{n!}A^n + ...$$

This operator is well-defined. Its properties are similar to those of the real-valued exponential function e^{at} for $a \in R$ and are proved in the same way.

THEOREM 10.19. If $A \in L(X)$ and $t \in R$, then the series in (10.17) converges uniformly in $L(X)$. Moreover, the function $t \mapsto e^{tA}$ belongs to $C^\infty(R; L(X))$ and for every $s, t \in R$

$$e^{sA}e^{tA} = e^{(s+t)A}, \frac{d}{dt}e^{tA} = Ae^{tA}.$$

Consider a linear homogeneous initial value problem

(10.18) $$u_t = Au, \qquad u(0) = f \in X \qquad u \in C^1(R; X)$$

The solution is given by the operator exponential.

THEOREM 10.20. The unique solution $u \in C^\infty(R, X)$ of (10.18) is given by

$$u(t) = e^{tA}f.$$

EXAMPLE 10.21. For $1 \le p < \infty$, let $A : L^p(R) \to L^p(R)$ be the bounded translation operator

$$Af(x) = f(x+1).$$

The solution $u \in C^\infty(R; L^p)$ of the differential-difference equation

$$u_t(x,t) = u(x+1,t), \quad u(x,0) = f(x)$$

Is given by

$$u(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f(x+n).$$

Example 10.22. Suppose that $a \in L^1(R^n)$ and define the bounded convolution operator $A : L^2(R^n) \to L^2(R^n)$ by $Af = a * f$. Consider the

IVP

$$u_t(x,t)=\int_{R^n} a(x-y)u(y)dy, \qquad u(x,0)=f(x)\in L^2(R^n).$$

Taking the Fourier transform of this equation and using the convolution Theorem, we get

$$u_t(k,t)=(2\pi)^n a^\wedge(k)u(k,t), \qquad u(k,0)=f^\wedge(k).$$

The solution is $u(k,t)=e^{(2\pi)^n a(k)t} f^\wedge(k)$.

It follows that

$$u(x,t)\int g(x-y,t)f(y)dy$$

Where the Fourier transform of $g(x,t)$ is given by

$$g^\wedge(k,t)=\frac{1}{(2\pi)}e^{(2\pi)na^\wedge(k)t}.$$

Since $a\in L^1(R^n)$, the Riemann-Lebesgue lemma implies that $a^\wedge \in C_0(R^n)$, and therefore $g^\wedge(.,t)\in C_b(R^n)$ for every $t\in N$. Since convolution with g corresponds to multiplication of the Fourier transform by a bounded multiplier, it defines a bounded linear amp on $L^2(R^n)$.

The solution operators $T(t)=e^{tA}$ of (10.18) form a uniformly continuous oneparameter group. Conversely, any uniformly contiuunous one-parameter group of transformations on a Banch space is generated by a bounded linear operator.

Definition 10.23. Let X be a Banach space. A one-parameter, uniformly continuous group on X is a family $\{T(t):t\in R\}$ of bounded linear operators $T(t): X\to X$ such that:

1) $T(0)=I$;

2) $T(s)T(t)=T(s+t)$ for all $s,t\in R$;

3) $T(h)\to I$ uniformly in $L(X)$ as $h\to 0$.

THEOREM 10.24. If $\{T(t):t\in R\}$ is a uniformly continuous group on a Banach space X, then:

1) $T \in C^{\infty}\left(R; L(X)\right)$;

2) $A = T_t(0)$ is a bounded linear operator on X;

3) $T(t) = e^{tA}$ for every $t \in R$.

Note that the differentiability (and, in fact, the analyticity) of $T(t)$ with respect to t is implied by its continuity and the group property $T(s)T(t) = T(s+t)$. This is analogous to what happens for the real exponential function: The only continuous functions $f : R \to R$ that satisfy the functional equation

(10.19) $f(0) = 1, \qquad f(s)f(t) = f(s+t)$ for all $s, t \in R$

are the exponential functions $f(t) = e^{at}$ for $a \in R$, and these functions are analytic.

Some regularity assumption on f is required in order for (10.19) to imply that f is an exponential function. If we drop the continuity assumption, then the function defined by $f(0) = 1$ and $f(t) = 0$ for $t \neq 0$ also satisfies (10.19). This function and the exponential functions are the only Lebesgue measurable solutions of (10.19). If we drop the measurability requirement, then we get many other solutions.

EXAMPLE 10.25. If $f = e^{g}$ where $g : R \to R$ satisfies

$$g(0) = 0, \qquad g(s) + g(t) = g(s+t).$$

then f satisfied (10.19). The linear functions $g(t) = at$ satisfy this functional equation for any $a \in R$, but there are many other non-measurable solutions. To "construct" examples, consider R as a vector space over the field R of rational numbers, and let $\{e_{\alpha} \in R : \alpha \in I\}$ denote an algebraic basis. Given any values $\{c_{\alpha} \in R : \alpha \in I\}$ define $g : R \to R$ such that $g(e_{\alpha}) = c_{\alpha}$ for each $\alpha \in I$, and if $x \sum x_{\alpha} e_{\alpha}$ is the finite expansion of $x \in R$ with respect to the basis, then

$$g\left(\sum x_\alpha e_\alpha\right) = \sum x_\alpha c_\alpha.$$

10.4.2. Strongly continuous semigroups. We may consider the heat equation and other linear evolution equations from a similar perspective to the Banach space ODEs discussed above. Significant differences arise, however, as a result of the fact that the Laplacian and other spatial differential operators are unbounded maps of a Banach space into itself. In particular, the solution operators associated with unbounded operators are strongly but not uniformly continuous functions of time, and we get solutions that are, in general, continuous but not continuously differentiable. Moreover, as in the case of the heat equation, we may only be able to solve the equation forward in time, which gives us a semigroup of solution operators instead of a group.

Abstracting the notion of a family of solution operators with continuous trajectories forward in time, we are led to the following definition.

Definition: 10.26. Let X be a Banach space. A one-parameter, strongly continuous (or C_0) semigroup on X is a family $\{T(t) : t \geq 0\}$ of bounded linear operators $T(t) : X \to X$ such that

1) $T(0) = I$;

2) $T(s)T(t) = T(s+t)$ for all $s, t \geq 0$;

3) $T(h)f \to f$ strongly in X as $h \to 0^+$ for every $f \in X$.

The semigroup is said to be a contraction semigroup if $\|T(t)\| \leq 1$ for all $t \geq 0$, where $\| \, . \, \|$ denotes the operator norm.

The semigroup property (2) holds for the solution maps of any well-posed au- tonomous evolution equation: it says simply that we can solve for time $s+t$ by solving for time t and then for time s. Condition (3) means explicitly that

$$\|T(t)f - f\|_X \to 0 \quad \text{as } t \to 0^+.$$

If this holds, then the semigroup property (2) implies that

$T(t+h)f \to T(t)f$ in X as $h \to 0$ for every $t > 0$, not only for $t = 0$ [8]. The term 'contraction' in Definition 10.26 is not used in a strict sense, and the norm of the solution of a contraction semigroup is not required to be strictly decreasing in time; it may for example, remain constant.

The heat equation

(10.20) $$u_t = \Delta u, \qquad u(x,0) = f(x)$$

is one of the primary motivating examples for the theory of semigroups. For definite- ness, we suppose that $f \in L^2$, but we could also define a heat-equation semigroup on other Hilbert or Banach spaces, such as H^s or L^p for $1 < p < \infty$.

From Theorem 10.12 with $s = 0$, for every $f \in L^2$ there is a unique generalized solution $u:[0,\infty) \to L^2$ of (10.20), and therefore for each $t \geq 0$ we may define a bounded linear map $T(t): L^2 \to L^2$ by $T(t): f \mapsto u(t)$. The operator $T(t)$ is defined explicitly by

(10.21) $$T(0) = I, \qquad T(t)f = \Gamma^t * f \text{ for } t > 0$$

$$T(t)f^{\wedge}(k) = e^{-t|k|^2} f^{\wedge}(k).$$

Where the * denotes spatial convolution with the Green's function $\Gamma^t(x) = \Gamma(x,t)$ given in

(10.6)

We also use the notation

$$T(t) = e^{t\Delta}$$

and interpret $T(t)$ as the operator exponential of $t\Delta$. The semigroup property then becomes the usual exponential formula

$$e^{(s+t)\Delta} = e^{s\Delta} e^{t\Delta}.$$

Theorem 10.27. *The solution operators* $\{T(t): t \geq 0\}$ *of the heat equation defined in (10.21) form a strongly continuous contraction semigroup on* $L^2(R^n)$.

Proof. This Theorem is a restatement of results that we have

Notes

already proved, but let us verify it explicitly. The semigroup property follows from the Fourier representation, since

$$e^{-(s+t)|k|^2} = e^{-s|k|^2} e^{-t|k|^2}.$$

It also follows from the spatial representation, since

$$\Gamma^{s+t} = \Gamma^s * \Delta^t.$$

The probabilistic interpretation of this identity is that the sum of independent Gaussian random variables is a Gaussian random variable, and the variance of the sum is the sum of the variances.

Theorem 10.12, with $s = 0$, implies that the semigroup is strongly continuous since $t \mapsto T(t) f$ belongs to $C([0,\infty); L^2)$ for every $f \in L^2$. Finally, it is immediate from (10.21) and Parseval's Theorem that $\|T(t)\| \le 1$ for every $t \ge 0$, so the semigroup is a contraction semigroup.

An alternative way to view this result is that the solution maps

$$T(t): S \subset L^2 \to S \subset L^2$$

constructed in Theorem 10.4 are defined on a dense subspace S of L^2, and are bounded on L^2, so they extend to bounded linear maps $T(t): L^2 \to L^2$ which form a strongly continuous semigroup.

Although for every $f \in L^2$ the trajectory $t \mapsto T(t) f$ is a continuous function from $[0,\infty)$ into L^2, it is *not* true that $t \mapsto T(t)$ is a continuous map from $[0,\infty)$ into the space $L(L^2)$ of bounded linear maps on L^2 since $T(t+h)$ does not converge to $T(t)$ as $h \to 0$ uniformly with respect to the operator norm.

but for f L^2 H^2 the solution is not differentiable with respect to t in L^2

Proposition 10.13 implies a solution $t \mapsto T(t) f$ belongs to $C^1([0,\infty); L^2)$. If $f \in H^2$, but for $f \in L^2 \setminus H^2$ the solution is not differentiable with respect to t in L^2 at $t = 0$. For every $t > 0$ however, we have from proposition 10.14 that the solution belongs to

$C^{\infty}\left(0,\infty;H^{\infty}\right)$. Thus, the the heat equation semiflow maps the entire phase space L^2 forward in time into a dense subspace H^{∞} of smooth functions. As a result of this smoothing, we cannot reverse the flow to obtain a map backward in time of L^2 into itself.

10.4.3. Strongly continuous groups. Conservative wave equations do not smooth solutions in the same way as parabolic equations like the heat equation, and they typically define a group of solution maps both forward and backward in time.

Definition 10.28. Let X be a Banach space. A one-parameter, strongly continuous (or C_0) group on X is a family $\{T(t): t \in R\}$ of bounded linear operators $T(t): X \to X$ such that

(1) $T(0) = I$;

(2) $T(s)T(t) = T(s+t)$ for all $s,t \in R$;

(3) $T(h)f \to f$ strongly in X as $h \to 0$ for every $f \in X$.

If X is a Hilbert space and each $T(t)$ is a unitary operator on X, then the group is said to be a unitary group.

Thus $\{T(t): t \in R\}$ is a strongly continuous group if and only if $\{T(t): t \geq 0\}$ is a strongly continuous semigroup of invertible operators and $T(-t) = T^{-1}(t)$.

Theorem 10.29. *Suppose that* $s \in R$. *The solution operators* $\{T(t): t \in R\}$ *of the Schrö̈dinger equation (10.13) defined by*

(10.22) $$\left(T(t)f^{\wedge}\right)(k) = e^{it|k|^2} f^{\wedge}(k).$$

form a strongly continuous, unitary group on $H^s\left(R^n\right)$.

Unlike the heat equation semigroup, the Schrodinger equation is a dispersive wave equation which does not smooth solutions. The solution maps $\{T(t): t \in R\}$ form a group of unitary operators on L^2 which map H^s onto itself (*c.f.* Theo- rem 10.17). A trajectory $u(t)$ belongs to $C^1\left(R;L^2\right)$ if and only if $u(0) \in H^2$ and $u \in C^k\left(R;L^2\right)$ if and only if $u(0) \in H^{1+k}$. If $u(0) \in L^2 \setminus H^2$, then

$u \in C\left(R : L^2\right)$ but u is nowhere strongly differentiable in L^2 with respect to time. Nevertheless, the continuous non-differentiable trajectories remain close in L^2 to the differentiable trajectories. This dense intertwining of smooth trajectories and continuous, non-differentiable trajectories in an infinite-dimensional phase space is not easy to imagine and has no analog for ODEs.

The Schrodinger operators $T(t) = e^{it\Delta}$ do *not* form a strongly continuous group on $L^p\left(R^n\right)$ when $p \neq 2$. Suppose, for contradiction, that $T(t) : L^p \to L^p$ is bounded for some $1 \leq p < \infty, p \neq 2$ and $t \in R \setminus \{0\}$. Then since $T(-t) = T^*(t)$, duality $T(-t) : L^{p'} \to L^{p'}$ is bounded, and we can assume that $1 \leq p < 2$ without loss of generality. From Theorem 10.16, $T(t) : L^p \to L^{p'}$ is bounded, and thus for every $f \in L^p \cap L^{p'} \subset L^2$

$$\| f \|_{L^p} = \left\| T(t) T(-t) f \right\|_{L^p} \leq C_1 \left\| T(-t) f \right\|_{L^{p'}} \leq C_1 C_2 \| f \| L^{p'}.$$

This estimate is false if $p \neq 2$, so $T(t)$ cannot be bounded on L^p.

10.4.4. Generators. Gi Generators. Given an operator A that generates a semigroup, we may define the semigroup $T(t) = e^{tA}$ as the collection of solution operators of the equation $u_t = Au$. Alternatively, given a semigroup, we may ask for an operator A that generates it.

Definition 10.30. Suppose that $\{T(t) : t \geq 0\}$ is a strongly continuous semi- group on a Banach space X. The generator A of the semigroup is the linear operator in X with domain $D(A)$,

$$A : D(A) \subset X \to X,$$

Defined as follows:

1) $f \in D(A)$ if and only if the limit

$$\lim_{h \to 0+} \left[\frac{T(h) f - f}{h} \right]$$

Exists with respect to the strong (norm) topology of X;

2) If $f \in D(A)$, then

$$Af = \lim_{h \to 0+} \left[\frac{T(h)f - f}{h} \right].$$

To des To describe which operators are generators of a semigroup, we recall some definitions and results from functional analysis. See [8] for further discussion and proofs of the results.

Definition 10.31. An operator $A : D(A) \subset X \to X$ in a Banach space X is closed if whenever $\{f_n\}$ is a sequence of points in $D(A)$ such that $f_n \to f$ and $Af_n \to g$ in X as $n \to \infty$, then $f \in D(A)$ and $Af = g$.

Equivalently, A is closed if its graph

$$G(A) = \{(f, g) \in X \times X : f \in D(A) \text{ and } Af = g\}$$

Is a closed subset of $X \times X$.

Theorem 10.32. *If A is the generator of a strongly continuous semigroup $\{T(t)\}$ on a Banach space X, then A is closed and its domain $D(A)$ is dense in X.*

Example 10.33. If $T(t)$ is the heat-equation semigroup on L^2, then the L^2-limit

$$\lim_{h \to 0+} \left[\frac{T(h)f - f}{h} \right]$$

exists if and only if $f \in H^2$, and then it is equal to Δf. The generator of the heat equation semigroup on L^2 is therefore the unbounded Laplacian operator with domain H^2,

$$\Delta : H^2(R^n) \subset L^2(R^n) \to L^2(R^n).$$

If $f_n \to f$ in L^2 and $\Delta f_n \to g$ in L^2, then the continuity of distributional derivatives implies that $\Delta f = g$ and elliptic regularity theory (or the explicit Fourier representation) implies that $f \in H^2$. Thus, the Laplacian with domain $H^2(R^n)$ is a closed operator in $L^2(R^n)$. It is also self-adjoint.

Not every closed, densely defined operator generates a semigroup: the powers of its resolvent must satisfy suitable estimates.

Definition 10.34. Suppose that $A: D(A) \subset X \to X$ is a closed linear operator in a Banach space X and $D(A)$ is dense in X. A complex number $\lambda \in C$ is in the resolvent set $\rho(A)$ of A if $\lambda I - A: D(A) \subset X \to X$ is one-to-one and onto. If $\lambda \in \rho(A)$, the inverse

(10.23)

$$R(\lambda, A) = (\lambda I - A)^{-1} : X \to X$$

is called the resolvent of A.

The open mapping (or closed graph) Theorem implies that if A is closed, then the resolvent $R(\lambda, A)$ is a bounded linear operator on X whenever it is defined. This is because $(f, Af) \mapsto \lambda f - Af$ is a one-to-one, onto map from the graph $G(A)$ of A to X, and $G(A)$ is a Banach space since it is a closed subset of the Banach space $X \times X$.

The resolvent of an operator A may be interpreted as the Laplace transform of the corresponding semigroup. Formally, if

$$u(\lambda) = \int_0^\infty u(t) e^{-\lambda t} dt$$

Is the Laplace transform of $u(t)$, then taking the Laplace transform with respect to t of the equation

$$u_t = Au \qquad u(0) = f,$$

We get

$$\lambda u - f = Au.$$

For $\lambda \in \rho(A)$, the solution of this equation is

$$u(\lambda) = R(\lambda, A) f.$$

This solution is the Laplace transform of the time-domain solution

$$u(t) = T(t) f$$

With $R(\lambda, A) = T(t)$, or

$$(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} e^{tA} dt.$$

This identity can be given a rigorous sense for the generators A of a semigroup, and it explains the connection between semigroups and resolvents. The Hille-Yoshida Theorem provides a necessary and sufficient condition on the resolvents for an operator to generate a strongly continuous semigroup.

Theorem 10.35. A linear operator $A: D(A) \subset X \to X$ in a Banach space X is the generator of a strongly continuous semigroup $\{T(t); t \geq 0\}$ on X if and only if there exist constants $M \geq 1$ and $a \in R$ such that the following conditions are satisfied:

1) The domain $D(A)$ is dense in X and A is closed;

2) Every $\lambda \in R$ such that $\lambda > a$ belongs to the resolvent set of A;

3) If $\lambda > a$ and $n \in R$, then

(10.24) $$\left\| R(\lambda, A)^n \right\| \leq \frac{M}{(\lambda - a)^n}$$

Where the resolvent $R(\lambda, A)$ is defined in (10.23).

In that case,

(10.25) $$\|T(t)\| \leq Me^{at} \quad \text{for all } t \geq 0.$$

This Theorem is often not useful in practice because the condition on arbitrary powers of the resolvent is difficult to check. For contraction semigroups, we have the following simpler version.

Corollary 10.36. *A linear operator* $A: D(A) \subset X \to X$ *in a Banach space* X *is the generator of a strongly continuous contraction semigroup* $\{T(t); t \geq 0\}$ *on* X *if and only if:*

(1) the domain $D(A)$ *is dense in* X *and* A *is closed;*

(2) every $\lambda \in R$ *such that* $\lambda > 0$ *belongs to the resolvent set of* A*;*

3) if $\lambda > 0$, *then*

(10.26) $$\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

Notes

This Theorem follows from the previous one since

$$\left\|R(\lambda,A)^n\right\| \le \left\|R(\lambda,A)\right\|^n \le \frac{1}{\lambda^n}.$$

The crucial condition here is that $M=1.$ we can always normalize $a=0$, since if A satisfies Theorem (10.35) with $a=\alpha,$ then $A-\alpha I$ satisfies Theorem (10.35) with $a=0$. Correspondingly, the substitution $u=e^{\alpha t}v$ transforms the evolution equation $u_t = Au$ to $v_t=(A-\alpha I)v.$

The Lumer-Phillips Theorem provides a more easily checked condition (that A is 'm-dissipative') for A to generate a contraction semigroup. This condition often follows for PDEs from a suitable energy estimate.

Definition 10.37. A closed, densely defined operator $A: D(A)\subset X \to X$ in a Banach space X is dissipative if for every $\lambda>0$

(10.27) $\quad \lambda\|f\| \le \|(\lambda I-A)f\| \quad$ for all $f\in D(A).$

The operator A is maximally dissipative, or $m-$ dissipative for short, if it is dissipative and the range of $\lambda I-A$ is equal to X for some $\lambda>0$.

The estimate (10.27) implies immediately that $\lambda I-A$ is one-to-one. It also implies that the range of $\lambda I-A: D(A)\subset X\to X$ is closed. To see this, suppose that g_n belongs to the range of $\lambda I-A$ and $g_n\to g$ in X. if $g_n=(\lambda I-A)f_n,$ then (10.27) implies that $\{f_n\}$ is Cauchy since $\{g_n\}$ is Cauchy, and therefore $f_n\to f$ for some $f\in X$. Since A is closed, it follows that $f\in D(A)$ and $(\lambda I-A)f=g$. Hence, g belongs to the range of $\lambda I-A$.

The range of $\lambda I-A$ may be a proper closed subspace of X for every $\lambda>0;$ if, however, A is $m-$ dissipative, so that $\lambda I-A$ is onto X for some $\lambda>0$. Then one can prove that $\lambda I-A$ is onto for every $\lambda>0$, meaning that the resolvent set of A contains the positive real axis $\{\lambda>0\}.$ The estimate (10.27) is then equivalent to (10.26). We therefore get the following result, called the Lumer-Phillips Theorem.

THEOREM 10.38. An operator $A : D(A) \subset X \to X$ ina Banch space X is the generator of a contraction semigroup on X if and only if:

1) A is closed and densely defined;

2) A is m-dissipative.

EXAMPLE 10.39. Consider $\Delta : H^2(R^n) \subset L^2(R^n) \to L^2(R^n)$. If $f \in H^2$, then using the integration-by-parts property of the weak derivative on H^2 we have for $\lambda > 0$ that

$$\|(\lambda I - \Delta) f\|_{L^2}^2 = \int (\lambda f - \Delta f)^2 \, dx$$

$$= \int \left[\lambda^2 f^2 - 2\lambda f \Delta f + (\Delta f)^2 \right] dx$$

$$= \int \left[\lambda^2 f^2 + 2\lambda Df.Df + (\Delta f)^2 \right] dx$$

$$\geq \lambda^2 \int f^2 dx.$$

Hence,

$$\lambda \|f\|_{L^2} \leq \|(\lambda I - \Delta) f\|_{L^2}$$

and Δ is dissipative. The range of $\lambda I - \Delta$ is equal to L^2 for any $\lambda > 0$, as one can see by use of the Fourier transform (in fact, $I - \Delta$ is an isometry of H^2 onto L^2). Thus, Δ is m-dissipative. The Lumer-Phillips Theorem therefore implies that $\Delta : H^2 \subset L^2 \to L^2$ generates a strongly continuous semigroup on $L^2(R^n)$, as we have seen explicitly by use of the Fourier transform.

Thus, in order to show that an evolution equation

$$u_t = Au \qquad u(0) = f$$

in a Banach space X generates a strongly continuous contraction semigroup, it is sufficient to check that $A : D(A) \subset X \to X$ is a closed, densely defined, dissipative operator and that for some $\lambda > 0$ the resolvent equation

$$\lambda f - Af = g$$

has a solution $f \in X$ for every $g \in X$.

EXAMPLE 10.40. The linearized Kuramoto-Sivashinsky (KS) equation is

$$u_t = -\Delta u - \Delta^2 u.$$

This equation models a system with long-wave instability, described by the back- ward heat-equation term $-\Delta u$, and short wave stability, described by the forth- order diffusive term $-\Delta^2 u$. The operator

$$A : H^4\left(R^n\right) \subset L^2\left(R^n\right) \to L^2\left(R^n\right), \qquad Au = \Delta u - \Delta^2 u$$

generates a strongly continuous semigroup on $L^2\left(R^n\right)$, or $H^s\left(R^n\right)$. One can verify this directly from the Fourier representation,

$$\left[e^{tA} f^\wedge\right](k) = e^{t\left(|k|^2 - |k|^4\right)} f^\wedge(k),$$

but let us check the hypotheses of the Lumer-Phillips Theorem instead. Note that

(10.28) $$|k|^2 - |k|^4 \le \frac{3}{16} \text{ for all } |k| \ge 0.$$

We claim that $A^\wedge = A - \alpha I$ is m-dissipative for $\alpha \le 3/16$. First, $A^\wedge$ is densely defined and closed, since if $f_n \in H^4$ and $f_n \to f, A^\wedge f_n \to g$ in L^2, the Fourier representation implies that $f \in H^4$ and $A^\wedge f = g$. If $f \in H^4$, then using (10.28), we have

$$\begin{aligned}\left\|\lambda f - A^\wedge f\right\|^2 &= \int_{\mathbb{R}^n} \left(\lambda + \alpha - |k|^2 + |k|^4\right)^2 \left|f^\wedge(k)\right|^2 dk \\ &\ge \lambda \int_{R^n} \left|f^\wedge(k)\right|^2 dk \\ &\ge \lambda \|f\|_{L^2}^2,\end{aligned}$$

Which means that $A^\wedge$ is dissipative. Moreover, $\lambda I - A^\wedge : H^4 \to L^2$ is one-to-one and onto for any $\lambda > 0$, since $\left(\lambda I - A^\wedge\right) f = g$ if and only if

$$f^\wedge(k) = \frac{g^\wedge(k)}{\lambda + \alpha - |k|^2 + |k|^4}.$$

Thus, $A^\wedge$ is m-dissipative, so it generates a contraction semigroup on L^2. It follows that A generates a semigroup on $L^2\left(R^n\right)$ such that

$$\left\|e^{tA}\right\|_{L\left(L^2\right)} \le e^{3t/16},$$

corresponding to $M = 1$ and $a = 3/16$ in (10.25).

Finally, we state Stone's Theorem, which gives an equivalence between self- adjoint operators acting in a Hilbert space and strongly continuous unitary groups. Before stating the Theorem, we give the definition of an unbounded self-adjoint operator. For definiteness, we consider complex Hilbert spaces.

DEFINITON 10.41. Let H be a complex Hilbert space with inner-product

$$(.,.): H \times H \to C.$$

An operator $A: D(A) \subset H \to H$ is self-adjoint if :

1) The domain $D(A)$ is dense in H;

2) $x \in D(A)$ if and only if there exists $z \in H$ such that $(x, Ay) = (z, y)$ for every $y \in D(A)$;

3) $(x, Ay) = (Ax, y)$ for all $x, y \in D(A)$

Condition (2) states that $D(A) = D(A^*)$ where A^* is the Hilbert space adjoint of A, in which case $z = Ax,$ while (3) states that A is symmetric on its domain. A precise characterization of the domain of a self-adjoint operator is essential; for differential operators acting in L^p-spaces, the domain can often be described by the use of Sobolev spaces. The next result is Stone's Theorem (see *e.g.* [44] for a proof).

Theorem 10.42. An operator $iA: D(iA) \subset H \to H$ in a complex Hilbert space H is the generator of a strongly continuous unitary group on H if and only if A is self-adjoint.

EXAMPLE 10.43. The generator of the Schrodinger group on $H^s(R^n)$ is the self- adjoint operator

$$i\Delta: D(i\Delta) \subset H^s(R^n) \to H^s(R^n), \qquad D(i\Delta) = H^{s+2}(R^n).$$

EXAMPLE 10.44. Consider the Klein-Gordon equation

$u_t = v, \;\; v_t = \Delta u,$

Which has the form $w_t = Aw$ where

Notes

$$w=\begin{pmatrix}u\\v\end{pmatrix},\qquad A=\begin{pmatrix}0 & I\\ \Delta-I & 0\end{pmatrix}.$$

We let

$$H=H^1\left(R^n\right)\otimes L^2\left(R^n\right)$$

With the inner product of $w_1-\left(u_1,v_1\right),w_2=\left(u_2,v_2\right)$ defined by

$$\left(w_1,w_2\right)_H=\left(u_1,u_2\right)_{H^1}+\left(v_1,v_2\right)_{L^2},\left(v_1,v_2\right)_{H^1}=\int\left(u_1u_2+Du_1.Du_2\right)dx.$$

Then the operator

$$A:D(A)\subset H\to H,\qquad D(A)=H^2\left(R^n\right)\oplus H^1\left(R^n\right)$$

Is self -adjoint and generates a unitary group on H.

We can instead take

$$H=L^2\left(R^n\right)\oplus H^{-1}\left(R^n\right),\qquad D(A)=H^1\left(R^n\right)\oplus L^2\left(R^n\right)$$

and get a unitary group on this larger space.

10.4.5. Nonhomogeneous equations. The solution of a linear nonhomogeneous ODE

(5.29) $$u_t=Au+g,\quad u(0)=f$$

may be expressed in terms of the solution operators of the homogeneous equation by the variation of parameters, or Duhamel, formula.

Theorem 5.45. *Suppose that* $A:X\to X$ *is a bounded* linear operator on a Banach space X and $T(t)=e^{tA}$ is the associated uniformly continuous group. If $f\in X$ and $g\in C(R;X)$ then the solution $u\in C^1(R;X)$ of (10.29) is given by

(10.30.) $$u(t)=T(t)f+\int_0^t T(t-s)g(s)ds.$$

This solution is continuously strongly differentiable and satisfies the ODE (10.29) pointwise in t for every $t\in$. We refer to such a solution as a classical solution. For a strongly continuous group with an unbounded generator, however, the Duhamel formula (10.30) need not define a function $u(t)$ that is differentiable at any time t even if $g\in C(R;X)$

EXAMPLE 10.46. Let $\{T(t):t\in R\}$ be a strongly continuous group on a Banach space X with generator $A:D(A)\supset X\to X$, and suppose that there exists $g_0\in X$ such that $T(t)g_0\notin D(A)$ for every $t\in\mathbb{R}$. For example, if $T(t)=e^{it\Delta}$ is the Schrodinger group on $L^2(R^n)$ and $g_0\notin H^2(R^n)$, then $T(t)g_0\notin H^2(R^n)$ for every $t\in R$. Taking $g(t)=T(t)g_0$ and $f=0$ in (10.30) and using the semigroup property, we get

$$u(t)=\int_0^t T(t-s)T(s)g_0 \qquad ds=\int_0^t T(t)g_0ds=tT(t)g_0.$$

This function is continuous but not differentiable with respect to t , since $T(t)f$ is differentiable at t_0 if and only if $T(t_0)f\in D(A)$.

It may happen that the function $u(t)$ defined in (10.30) is is differentiable with respect to t in a distributional sense and satisfies (10.29) pointwise almost everywhere in time. We therefore introduce two other notions of solution that are weaker than that of a classical solution.

Definition :10.47. Suppose that A be the generator of a strongly continuous semigroup $\{T(t):t\geq 0\}$, $f\in X$ and $g\in L^1([0,T];X)$. A function $u:[0,T]\to X$ is a strong solution of (10.29) on $[0,T]$ if:

1) u is absolutely continuous on $[0,T]$ with distributional derivative $u_t\in L^1(0,T;x)$;

2) $u(t)\in D(A)$ pointwise almost everywhere for $t\in(0,T)$;

3) $u_t(t)=Au(t)+g(t)$ pointwise almost everywhere for $t\in(0,T)$;

4) $u(0)=f$.

A function $u:[0,T]\to X$ is a mild solution of (10.29) on $[0,T]$ if u is given by (10.30) for $t\in[0,T]$.

Notes

Every classical solution is a strong solution and every strong solution is a mild solution. As example (10.46) shows, however, a mild solution need not be a strong solution.

The Duhamel formula provides a useful way to study semilinear evolution equations of the form

(10.31) $$u_t = Au + g(u)$$

where the linear operator A generates a semigroup on a Banach space X and

$$g : D(F) \subset X \to X$$

is a nonlinear function. For semilinear PDEs, $g(u)$ typically depends on u but none of its spatial derivatives and then (10.31) consists of a linear PDE perturbed by a zeroth-order nonlinear term.

If $\{T(t)\}$ is the semigroup generated by A, we may replace (10.31) by an integral

equation for $u : [0, T] \to X$

(10.32) $$u(t) = T(t)u(0) + \int_0^t T(t-s)\, g(u(s))ds.$$

We then try to show that solutions of this integral equation exist. If these solutions have sufficient regularity, then they also satisfy (10.31).

In the standard Picard approach to ODEs, we would write (10.31) as

(10.33) $$u(t) = u(0) + \int_0^t \left[Au(s) + g(u(s)) \right] ds.$$

The advantage of (10.32) over (10.33) is that we have replaced the unbounded operator A by the bounded solution operators $\{T(t)\}$. Moreover, since $T(t-s)$ acts on $g(u(s))$ it is possible for the regularizing properties of the linear operators T to compensate for the destabilizing effects of the nonlinearity F. For example, in Section 10.5 we study a semilinear heat equation, and in Section 10.6 to prove the existence of solutions of a nonlinear Schrodinger equation.

10.10 NON-AUTONOMOUS EQUATIONS

The semigroup property $T(s)T(t)=T(s+t)$ holds for autonomous evolution equations that do not depend explicitly on time. One can also consider time-dependent linear evolution equations in a Banach space X of the form

$$u_t = A(t)u$$

where $A(t):D(A(t))\subset X \to X$. The solution operators $T(t;s)$ from time s to time t of a well-posed nonautonomous equation depend separately on the initial and final times, not just on the time difference; they satisfy

$$T(t,s)T(s;r)=T(t;r) \qquad \text{for} \quad r\le s\le t.$$

The time-dependence of A makes such equations more difficult to analyze fromthe semigroup viewpoint than autonomous equations. First, since the domain of $A(t)$ depends in general on t, one must understand how these domains are related and for what times a solution belongs to the domain. Second, the operators $A(s), A(t)$ may not commute for $s\neq t$, meaning that one must order them correctly with respect to time when constructing solution operators $T(t;s)$.

Similar issues arise in using semigroup theory to study quasi-linear evolution
equations of the form

$$u_t = A(u)u$$

in which, for example, $A(u)$ is a differential operator acting on u whose coefficients depend on u (see e.g. [44] for further discussion). Thus, while semigroup theory is an effective approach to the analysis of autonomous semilinear problems, its application to nonautonomous or quasilinear problems often leads to considerable technical difficulties.

10.11 LET US SUM UP

In this unit we have discussed about Heat equation, Schwartz solution, Irreversibility, Generalized solutions, The Schrodinger equation, Semi groups and groups, Non-autonoums equations. The heat, or diffusion, equation is $u_t = \Delta u.$ Steady solutions of the heat equation satisfy Laplace's equation. Suppose That $u \in C(a,b;S)$ where $u(t) = u(.,t)$. Then $u \in C^1(a,b;S)$ if and only if:

1. The pointwise partial derivative $\partial_t u(x,t)$ exists for every $x \in R^n$ and $t \in (a,b;)$

2. $\partial_t u(.,t) \in S$ for every $t \in (a,b)$;

3. The map $t \mapsto \partial_t u(.,t)$ belongs $C(a,b;S)$.

A solution $u(x,t)$ of the initial value problem for the heat equation on R^n is not unique without the imposition of a suitable growth condition as $|x| \to \infty$.

10.12 KEY WORDS

1. The heat, or diffusion, equation is $u_t = \Delta u.$
2. Solutions of the heat equation satisfy Laplace's equation.
3. Schrodinger equation $iu_t = -\Delta u$
4. PDE is a dispersive wave equation, which describes a complex wave-field that oscillates with a frequency proportional to the difference between the value of the function and its nearby means.

5.Suppose That $u \in C(a,b;S)$ where $u(t) = u(.,t)$. Then $u \in C^1(a,b;S)$ if and only if:

4. The pointwise partial derivative $\partial_t u(x,t)$ exists for every $x \in R^n$ and $t \in (a,b;)$

5. $\partial_t u(.,t) \in S$ for every $t \in (a,b)$;

3. The map $t \mapsto \partial_t u(.,t)$ belongs $C(a,b;S)$.

6. An operator $iA : D(iA) \subset H \to H$ in a complex Hilbert space H is the generator of a strongly continuous unitary group on H if and only if A is self-adjoint.

10.13 QUESTIONS FOR REVIEW

1. Discuss about Heat equation and Laplace equation
2. Discuss about Schwartz solutions
3. Discuss about the Schrodinger equation
4. Discuss about Semi groups and groups
5. Discuss about Non-autonoums equations

10.14 SUGGESTED READINGS AND REFERENCES

1. S. L. Ross, Differential Equations, 3rd Edn., Wiley India, 1984.

2. DiBenedetto, Partial Differential Equations, Birkhaüser, 1995.

3. L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.

4. I.N. Sneddon Elements of Partial Differential Equations McGrawHill 1986.

5. R. Churchil & J. Brown, Fourier Series & Boundary Value Problems.

6. R.C. McOwen , Partial Differential Equations (Pearson Edu.) 2003.

7. Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum, Outline Series, McGraw hill Series.

8. Partial Differential Equations, -Walter A.Strauss

9. Partial Differential Equations,-John K.Hunter

10. Partial Differential Equations,*Erich Mieremann*

11. Partial Differential Equations,-*Victor Ivrii*

10.15 ANSWERS TO CHECK YOUR PROGRESS

1. See Section 10.2
2. See Section 10.4
3. See Section 10.4
4. See Section 10.4
5. See Section 10.7

UNIT-11 THE HEAT AND SCHRODINGER EQUATIONS PART-2

STRUTURE

11.0 OBJECTIVE

In this unit we will learn about A semi linear equation, The non linear Schrodinger equations, Strichartz estimates, Local L^2-Solutions, The fourier transforms, The Sobolev spaces and Fractional integrals.

11.1 INTRODUCTION

In this unit we will dicsuss about negative Laplacian operator, The Schwartz space and The fourier transforms on S,The fourier transforms on S^1, The fourier transforms on L^1, The fourier transforms on L^2,The fourier transforms on L^p.

11.2 A SEMI LINEAR HEAT EQUATION

Consider the following initial value problem for $u : R^n \times [0,T] \to R$:

(11.34) $$u_t = \Delta u + \lambda u - \gamma u^m, \quad u(x,0) = g(x)$$

where $\lambda, \gamma \in R$ and $m \in R$ are parameters. This PDE is a scalar, semilinear reaction diffusion equation. The solution $u = 0$ is linearly stable when $\lambda < 0$ and linearly unstable when $\lambda > 0$. The nonlinear reaction term is potentially stabilizing if $\gamma > 0$ and m is odd or m is even and solutions are nonnegative (they remain nonegative by the maximum principle). For example, if $m = 3$ and $\gamma > 0$, then the spatially-independent reaction ODE $u_t = \lambda u - \gamma u^3$ has a supercritical pitchfork bifurcation at $u = 0$ as λ passes through 0. Thus, (11.34) provides a model equation for the study of bifurcation and loss of stability of equilbria in PDEs.

We consider (11.34) on Rn since this allows us to apply the results obtained earlier in the Chapter for the heat equation on R^n. In some respects, the behavior this IBVP on a bounded domain is simpler to analyze. The negative Laplacian on R^n does not have a compact resolvent and has a purely continuous spectrum $[0,\infty)$. By contrast, negative Laplacian on a bounded domain, with say homogeneous Dirichlet boundary conditions, has compact resolvent and a discrete set of eigenvalues $\lambda_1 < \lambda_2 \le \lambda_3 \le ......$ As a result, only finitely many modes become unstable as λ increases, and the long time dynamics of (11.34) is essentially finite-dimensional in nature.

Equations of the form

$$u_t = \Delta u + f(u)$$

on a bounded one-dimensional domain were studied by Chafee and Infante (1974), so this equation is sometimes called the Chafee-Infante equation. We consider here the special case with

$$f(u) = \lambda u - \gamma u^m \tag{11.35}$$

so that we can focus on the essential ideas. We do not attempt to obtain an optimal result; our aim is simply to illustrate how one can use semigroup theory to prove the existence of solutions of semilinear parabolic equations such as (11.34). Moreover, semigroup theory is not the only possible approach to such problems. For example, one can also use a Galerkin method.

11.2.1. Note

We will use the linear heat equation semigroup to reformulate (11.34) as a nonlinear integral equation in an appropriate function space and apply a contraction mapping argument.

To motivate the following analysis, we proceed formally at first. Suppose that $A = -\Delta$ generates a semigroup e^{-tA} on some space X, and let F be the nonlinear operator $F(u) = f(u)$, meaning that F is composition with f regarded as an operator on functions. Then (11.34) maybe written as the abstract evolution equation

$$u_t = -Au + F(u), \qquad u(0) = g.$$

Using Duhamel's formula, we get

$$u(t) = e^{-tA}g + \int_0^t e^{-(t-s)A} F(u(s))\,ds.$$

We use this integral equation to define mild solutions of the equation.

We want to formulate the integral equation as a fixed point problem $u = \Phi(u)$ on a space of Y-valued functions $u:(0,T) \to Y$. There are many ways to achieve this. In the framework we use here, we choose spaces $Y \subset X$ such that: (a) $F : Y \to X$ is locally Lipschitz continuous;

(b) $e^{-tA}: X \to Y$ for $t > 0$ with integrable operator norm as $t \to 0^+$. This allows the smoothing of the semigroup to compensate for a loss of regularity in the nonlinearity.

As we will show, one appropriate choice in $1 \le n \le 3$ space dimensions is $X = L^2(R^n)$ and $Y = H^{2\alpha}(R^n)$ for $n/4 < \alpha < 1$. Here $H^{2\alpha}(R^n)$ is the L^2 -Sobolev space of fractional order 2α defined in Section 11.C. We write the order of the Sobolev space as 2α because $H^{2\alpha}(R^n) = D(A^\alpha)$ is the domain of the α th-power of the generator of the semigroup.

11.2.2. Mild solutions

Let A denote the negative Laplacian operator in L^2,

(11.36)

$$A: D(A) \subset L^2(R^n) \to L^2(R^n), \qquad A = -\Delta, \qquad D(A) = H^2(R^n).$$

We define A as an operator acting in L^2 because we can study it explicitly by use of the Fourier transform.

As discussed in Section 11.4.2, A is a closed, densely defined positive operator, and $-A$ is the generator of a strongly continuous contraction semigroup

$$\{e^{-tA} : t \ge 0\}$$

on $L^2(R^n)$. The Fourier representation of the semigroup operators is

(11.37)

$$e^{-tA}: L^2(R^n) \to L^2(R^n), \qquad (e^{-tA}h^\wedge)(k) = e^{-t|k|^2} h(k).$$

If $t > 0$ we have for any $\alpha > 0$ that

$$e^{-tA}: L^2(R^n) \to H^{2\alpha}(R^n).$$

This property expresses the instantaneous smoothing of solutions of the heat equation c.f. Proposition 11.14.

We define the nonlinear operator

(11.38)

$$F: H^{2\alpha}(R^n) \to L^2(R^n), \qquad F(h)(x) = \lambda h(x) - \gamma h^m(x).$$

In order to ensure that F takes values in L^2 and has good continuity properties, we assume that $\alpha > n/4$. The Sobolev embedding theorem (Theorem 11.79) implies that $H^{2\alpha}\left(R^n\right) \to C_0\left(R^n\right)$. Hence, if $h \in H^{2\alpha}$, , then $h \in L^2 \cap C_0$, so $h \in L^p$ for every $2 \le p \le \infty$, and $F\left(h\right) \in L^2 \cap C_0$. We then define mild $H^{2\alpha}$ -valued solutions of (11.34) as follows.

DEFINITION 11.48. Suppose that $T > 0, \alpha > n/4$, and $g \in H^{2\alpha}\left(R^n\right)$. A mild $H^{2\alpha}$ -valued solution of (11.34) on $\left[0,T\right]$ is a function

$$u \in C\left(\left[0,T\right]; H^{2\alpha}\left(R^n\right)\right)$$

such that

(11.39)
$$u\left(t\right) = e^{-tA}g + \int_o^t e^{-(t-s)A} F\left(u\left(s\right)\right) ds$$

for every $0 \le t \le T$,

where e^{-tA} is given by (11.37), and F is given by (11.38).

11.3.3. Existence.

In order to prove a local existence result, we choose α large enough that the nonlinear term is well-behaved by Sobolev embedding, but small enough that the norm of the semigroup maps from L^2 into $H^{2\alpha}$ is integrable as $t \to 0^+$. As we will see, this is the case if $n/4 < \alpha < 1$, so we restrict attention to $1 \le n \le 3$ space dimensions.

THEOREM 11.49. Suppose that $1 \le n \le 3$ and $n/4 < \alpha < 1$. Then there exists $T > 0$, depending only on $\alpha, n, \|g\| H^{2\alpha}$, and the coefficients of f, such that (11.34) has a unique mild solution $u \in C\left(\left[0,T\right]; H^{2\alpha}\right)$ in the sense of Definition 11.48.

PROOF. We write (11.39) as

$$u = \Phi\left(u\right),$$

(11.40)
$$\Phi : C\left(\left[0,T\right]; H^{2\alpha}\right) \to C\left(\left[0,T\right]; H^{2\alpha}\right),$$

$$\Phi\left(u\right)\left(t\right) = e^{-tA}g + \int_0^t e^{-(t-s)A} F\left(u\left(s\right)\right) ds.$$

We will show that Φ defined in (11.40) is a contraction mapping on a suitable ball in $C\left(\left[0,T\right]; H^{2\alpha}\right)$. We do this in a series of Lemmas. The first Lemma is an estimate of the norm of the semigroup operators on the domain of a fractional power of the generator.

LEMMA 11.50. Let e^{-tA} be the semigroup operator defined in (11.37) and $\alpha > 0$. If t > 0, then

$$e^{-tA} : L^2\left(R^n\right) \to H^{2\alpha}\left(R^n\right)$$

and there is a constant $C = C\left(\alpha, n\right)$ such that

$$\left\|e^{-tA}\right\|_{L\left(L^2, H^{2\alpha}\right)} \le \frac{Ce^t}{t^\alpha}.$$

PROOF. Suppose that $h \in L^2\left(R^n\right)$. Using the Fourier representation

(11.37) of e^{-tA} as multiplication by $e^{-t|k|^2}$ and the definition of the $H^{2\alpha}$-norm, we get that

$$\left\|e^{-tA}h\right\|_{H^{2\alpha}}^2 = \left(2\pi\right)^n \int_{\mathbb{R}^n} \left(1+|k|^2\right)^{2\alpha} e^{-2t|k|^2} \left|h\left(k\right)\right|^2 dk$$

$$\le \left(2\pi\right)^n \sup_{k\in\mathbb{R}^n} \left[\left(1+|k|^2\right)^{2\alpha} e^{-2t|k|^2}\right] \int_{\mathbb{R}^n} \left|h\left(k\right)\right|^2 dk.$$

Hence, by Parseval's theorem,

$$\left\|e^{-tA}h\right\|_{H^{2\alpha}} \le M \left\|h\right\|_{L^2}$$

Where

$$M = \left(2\pi\right)^{n/2} \sup_{k\in\mathbb{R}^n} \left[\left(1+|k|^2\right)^{2\alpha} e^{-2t|k|^2}\right]^{1/2}.$$

Writing $1+|k|^2 = x,$ we have

$$M = \left(2\pi\right)^{n/2} e^t \sup_{x\ge 1} \left[x^\alpha e^{-tx}\right] \le \frac{Ce^t}{t^\alpha}.$$

And the result follows.

Next, we show that Φ is a locally Lipschitz continuous map on the space $C\left(\left[0,T\right]; H^{2\alpha}\left(R^n\right)\right).$

LEMMA 11.51. Suppose that $\alpha > n/4.$ Let Φ be the map defined in (11.40) where F is given by (11.38), A is given by (11.36) and $g \in H^{2\alpha}\left(R^n\right).$ Then

(11.41) Φ :

$$C\left(\left[0,T\right]; H^{2\alpha}\left(R^n\right)\right) \to C\left(\left[0,T\right]; H^{2\alpha}\left(R^n\right)\right)$$

and there exists a constant $C = C(\alpha, m, n)$ such that

$$\left\|\Phi(u) - \Phi(v)\right\|_{C([0,T];H^{2\alpha})}$$

$$\le CT^{1-\alpha}\left(1 + \|u\|^{m-1}_{C([0,T];H^{2\alpha})} + \|v\|^{m-1}_{C([0,T];H^{2\alpha})}\right)\|u - v\|_{C([0,T];H^{2\alpha})}$$

for every $u, v \in C([0,T]; H^{2\alpha})$.

ROOF. We write Φ in (11.40) as

$$\Phi(u)(t) = e^{-tA}g + \Psi(u)(t), \qquad \Psi(u)(t) = \int_0^t e^{-(t-s)A} F(u(s))ds.$$

Since $g \in H^{2\alpha}$ and $\{e^{-tA} : t \ge 0\}$ is a strongly continuous semigroup on $H^{2\alpha}$, the map $t \mapsto e^{-tA}g$ belongs to $C([0,T]; H^{2\alpha})$. Thus, we only need to prove the result for Ψ .

The fact that $\Psi(u) \in C([0,T]; H^{2\alpha})$ if $u \in C([0,T]; H^{2\alpha})$ follows from the Lipschitz continuity of Ψ and a density argument. Thus, we only need to prove the Lipschitz estimate.

If $u, v \in C([0,T]; H^{2\alpha})$, then using Lemma 11.50 we find that

$$\left\|\Phi(u)(t) - \Psi(u)(t)\right\|_{H^{2\alpha}} \le C\int_0^t \frac{e^{-(t-s)A}}{|t-s|^{\alpha}} \left\|F(u(s)) - F(u(s))\right\|_{L^2} ds$$

$$\le C \sup_{0 \le s \le T} \left\|F(u(s)) - F(u(s))\right\|_{L^2} \int_0^t \frac{1}{|t-s|^{\alpha}} ds.$$

Evaluating the s-integral, with $\alpha < 1$, and taking the supremum of the result over $0 \le t \le T$, we get

(11.42)

$$\left\|\Psi(u) - \Psi(v)\right\|_{L^\infty(0,T;H^{2\infty})} \le CT^{1-\alpha}\left\|F(u) - F(v)\right\|_{L^\infty(0,T;H^2)}.$$

From (11.35), if $g, h \in C_0 \subset H^{2\alpha}$ we have

$$\left\|F(g) - G(h)\right\|_{L^2} \le |\lambda| \|g - h\|_{L^2} + |\gamma| \left\|g^m - h^m\right\|_{L^2}$$

And

$$\left\|g^m - h^m\right\|_{L^2} \le C\left(\|g\|^{m-1}_{L^\infty} + \|h\|^{m-1}_{L^\infty}\right)\|g - h\|_{L^2}.$$

Hence, using the Sobolev inequality $\|g\|_{L^\infty} \le C\|g\|_{H^{2\alpha}}$ for $\alpha > n/4$ and the fact that $\|g\|_{L^2} \le \|g\|_{H^{2\alpha}}$, we get that

$$\left\|F(g)-F(h)\right\|_{L^2} \le C\left(1+\|g\|_{L^\infty}^{m-1}+\|h\|_{H^{2\infty}}^{m-1}\right)\|g-h\|_{H^{2\alpha}},$$

which means that $F: H^{2\alpha} \to L^2$ is locally Lipschitz continuous. The use of this result in (11.42) proves the Lemma.

Actually, under the assumptions we make here, $F: H^{2\alpha} \to H^{2\alpha}$ is locally Lipschitz continuousas a map from $H^{2\alpha}$ into itself, and we don't need to use the smoothing properties of the heat equation semigroup to obtain a fixed point problem in $C\left([0,T];H^{2\alpha}\right)$, so perhaps this wasn't the best example to choose! For stronger nonlinearities, however, it would be necessary to use the smoothing.

The existence theorem now follows by a standard contraction mapping argument. If $\|g\|_{H^{2\alpha}} = R,$ then

$$\left\|e^{-tA}\right\|_{H^{2\alpha}} \le R \text{ for every } 0 \le t \le T$$

since $\left\{e^{-tA}\right\}$ is a contraction semigroup on $H^{2\alpha}$. Therefore, if we choose

$$E=\left\{u \in C\left([0,T]\right); H^{2\alpha} : \|u\|_{C\left([0,T];H^{2\alpha}\right)} 2R\right\}$$

we see from Lemma 11.51 that $\Phi: E \to E$ if we choose $T>0$ such that

$$CT^{1-\alpha}\left(1+2R^{m-1}\right)=\theta R$$

Where $0<\theta<1.$ Moreover, in that case

$$\left\|\Phi(u)-\Phi(v)\right\|_{C\left([0,T];H^{2\alpha}\right)} \le \theta\|u-v\|_{C\left([0,T];H^{2\alpha}\right)} \qquad \text{for every } u,v \in E.$$

The contraction mapping theorem then implies the existence of a unique solution $u \in E.$

This result can be extended and improved in many directions. In particular, if A is the negative Laplacian acting in $L^p\left(R^n\right).$

$$A: W^{2,p}\left(R^n\right) \subset L^p\left(R^n\right) \to L^p\left(R^n\right), \qquad A=-\Delta.$$

then one can prove that $-A$ is the generator of a strongly continuous semigroup on L^p for every $1<p<\infty.$ Moreover, we can define fractional powers of A

$$A^\alpha: D\left(A^\alpha\right) \subset L^p\left(R^n\right) \to L^p\left(R^n\right).$$

If we choose $2p>n$ and $n/2p>\alpha<1,$ then Sobolev embedding implies that $D\left(A^\alpha\right) \to C_0$ and the same argument as the one above applies. This gives the existence of local mild solutions with values in $D\left(A^{\alpha}\right)$ in any number of space dimensions. The proof of the necessary estimates and embedding theorems is more involved that the proofs above if $p \neq 2$, since we cannot use the Fourier transform

to obtain out explicit solutions.

More generally, this local existence proof extends to evolution equations of the form ([41], §111.1)

$$u_t + Au = F(u),$$

where we look for mild solutions $u \in C\left(\left[0,T\right]; X\right)$ taking values in a

Banach space

X and there is a second Banach spaces Y such that:

(1) $e^{-tA} : X \to X$ is a strongly continuous semigroup for $t \geq 0$;

(2) $F : X \to Y$ is locally Lipschitz continuous;

(3) $e^{-tA} : Y \to X$ for $t > 0$ and for some $\alpha < 1$

$$\left\|e^{-tA}\right\|_{L(X,Y)} \leq \frac{C}{t^{\alpha}} \text{ for } 0 < t \leq T.$$

In the above example, we used $X = H^{2\alpha}$ and $Y = L^2$. If A is a sectorial operator that generates an analytic semigroup on Y, then one can define fractional powers A^{α} of A, and the semigroup $\left\{e^{-tA}\right\}$ satisfies the above properties with $X = D\left(A^{\alpha}\right)$ for $0 \leq \alpha < 1$ [36]. Thus, one gets a local existence result provided that $F : D\left(A^{\alpha}\right) \to L^2$ is locally Lipschitz, with an existence-time that depends on the X -norm of the

initial data.

In general, the X -norm of the solution may blow up in finite time, and one gets only a local solution. If, however, one has an a priori estimate for $\left\|u(t)\right\|_X$ that is global in time, then global existence follows from the local existence result.

Check your progress

1. Explain about a semi linear heat equation.

.--

11.3 THE NONLINEAR SCHRODINGER EQUATION

The nonlinear Schrodinger (NLS) equation is

(11.43) $$iu_t = -\Delta u - \lambda |u|^{\alpha} u$$

where $\lambda \in R$ and $\alpha > 0$ are constants. In many applications, such as the asymptotic description of weakly nonlinear dispersive waves, we get $\alpha = 2$, leading to the cubically nonlinear NLS equation.

A physical interpretation of (11.43) is that it describes the motion of a quantum mechanical particle in a potential $V = -\lambda |u|^{\alpha}$ which depends on the probability density $|u|^2$ of the particle c.f. (11.14). If $\lambda \neq 0$ we can normalize $\lambda = \pm 1$ so the magnitude of λ is not important; the sign of λ is, however, crucial.

If $\lambda > 0$, then the potential becomes large and negative when $|u|^2$ becomes large, so the particle 'digs' its own potential well; this tends to trap the particle and further concentrate is probability density, possibly leading to the formation of singularities in finite time if $n \geq 2$ and $\alpha \geq 4/n$. The resulting equation is called
the focusing NLS equation.

If $\lambda < 0$, then the potential becomes large and positive when $|u|^2$ becomes large; this has a repulsive effect and tends to make the probability density spread out. The resulting equation is called the defocusing NLS equation. The local L^2-existence result that we obtain here for subcritical nonlinearities $0 < \alpha < 4/n$ is,
however, not sensitive to the sign of λ.

The one-dimensional cubic NLS equation

$$iu_t + u_{xx} + \lambda |u|^2 u = 0$$

is completely integrable. If $\lambda > 0$, this equation has localized traveling wave solutions called solitons in which the effects of nonlinear self-focusing balance the tendency of linear dispersion to spread out the the wave. Moreover, these solitons preserve their identity under nonlinear interactions with other solitons. Such localized solutions exist for the focusing NLS equation in higher dimensions, but the NLS equation is not integrable if $n \geq 2$, and in that case the soliton solutions are not preserved under nonlinear interactions.

In this section, we obtain an existence result for the NLS equation. The linear Schrodinger equation group is not smoothing, so we cannot use it to compensate for the nonlinearity at a fixed time as we did in Section 11.5 for the semilinear equation. Instead, we use some rather delicate space-time estimates for the linear Schrodinger equation, called Strichartz estimates, to recover the powers lost by the nonlinearity.

We derive these estimates first.

11.3.1. Strichartz estimates.

The Strichartz estimates for the Schrodinger equation (11.13) may be derived by use of the interpolation estimate in Theorem 11.16 and the Hardy-Littlewood-Sobolev inequality in Theorem 11.77. The space-time norm in the Strichartz estimate is $L^q(R)$ in time and $L^r(R^n)$ in space for suitable exponents (q, r), which we call an admissible pair.

Definition 11.52. The pair of exponents (q, r) is an admissible pair if

$$\frac{2}{q} = \frac{n}{2} - \frac{n}{r}$$

Where $2 < q < \infty$ and

(11.45) $$2 < r < \frac{2n}{n-2} \quad \text{if } n \geq 3$$

Or $2 < r < \infty$ if $n = 1, 2$.

The Strichartz estimates continue to hold for some endpoints with $q = 2$ or $q = \infty$, but we will not consider these cases here.

THEOREM 11.53. Suppose that $\{T(t) : t \in R\}$ is the unitary group of solution operators of the Schrodinger equation on R^n defined in (11.22) and (q, r) is an admissible pair as in Definition 11.52.

(1) For $f \in L^2(R^n)$, let $u(t) = T(t) f$. Then $u \in L^q(R; L^r)$, and there is a

constant $C(n, r)$ such that

(11.46) $\|u\|_{L^q(R;L^r)} \le C\|f\|_{L^2}.$

(2) for $g \in L^{q'}\left(R;L^{r'}\right)$, Let

$$v(t) = \int_{-\infty}^{\infty} T(t-s)\,g(s)\,ds.$$

Then $v \in L^{q'}\left(R;L^{r'}\right) \cap C\left(R;L^2\right)$ and there is a constant $C(n,r)$ such that

(11.47) $$\|v\|_{L^\infty(R;L^2)} \le C\|g\|_{L^{q'}\left(R;L^{r'}\right)},$$

(11.48) $$\|v\|_{L^q(R;L^r)} \le C\|g\|_{L^{q'}\left(R;L^{r'}\right)}.$$

Proof. By a density argument, it is sufficient to prove the result for smooth functions. We therefore assume that $g \in C_c^\infty(R;S)$ is a smooth Schwartz-valued function with compact support in time and $f \in S.$ We prove the inequalities in
reverse order.
Using the interpolation estimate Theorem 11.16, we have for $2 < r < \infty$ that

$$\|v(t)\|_{L^r} \le \int_{-\infty}^{\infty} \frac{\|g(s)\|_{L^{r'}}}{\left(4\pi|t-s|\right)^{n(1/2-1/r)}}ds.$$

If r is admissible, then $0 < n(1/2 - 1/r) < 1.$ Thus, taking the L^q-norm of this inequality with respect to t and using the Hardy-Littlewood-Sobolev inequality (Theorem 11.77) in the result, we find that

$$\|v\|_{L^q(R;L^r)} \le C\|g\|_{L^q\left(R;L^{r'}\right)}$$

Where p is given by

$$\frac{1}{p} = 1 + \frac{1}{q} + \frac{n}{r} - \frac{n}{2}.$$

If q, r satisfy (11.44), then $p = q'$, and we get (11.48).

Using Fubini's theorem and the unitary group property of $T(t)$, we have

$$\left(v(t), v(t)\right)_{L^2(R^n)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(T(t-r)g(r), T(t-s)g(s)\right)_{L^2(R^n)} drds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(T(s-r)g(r), g(s)\right)_{L^2(R^n)} drds$$

$$= \int_{-\infty}^{\infty} \left(v(s), g(s)\right)_{L^2(R^n)} ds.$$

Using Holder's inequality and (11.48) in this equation, we get

$$\|v(t)\|_{L^2(R^n)}^2 \le \|v\|_{L^q(R;L^r)} \|g\|_{L^q(R;L^r)} \le C\|g\|_{L^{q'}(R;L^{r'})}^2.$$

Taking the supremum of this inequality over $t \in R$, we obtain (11.47). In fact, since

$$v(t) = T(t)\int_{-\infty}^{\infty} T(-s)\, g(s)\, ds$$

$v \in C(R;L^2)$ is and L^2 – solution of the homogeneous Schrodinger equation and $\|v(t)\|_{L^2(R^n)}$ is independent of t.

If $f \in S, u(t) = T(t) f$, and $g \in C_c^\infty(R,s)$, then using (11.48) we get

$$\int_{-\infty}^{\infty} (u(t), g(t))_{L^2} dt = \int_{-\infty}^{\infty} (T(t) f, g(t))_{L^2} dt$$

$$= \left(f, \int_{-\infty}^{\infty} T(t) f, g(t)\, dt \right)_{L^2}$$

$$\le \| f\|_{L^2} \left\| \int_{-\infty}^{\infty} T(-t)\, g(t)\, dt \right\|_{L^2}$$

$$\le C\| f\|_{L^2} \|g\|_{L^{q'}(R;L^{r'})}.$$

It then follows by duality and density that

$$\|u\|_{L^q(R;L^r)} = \sup_{g\in C_c^\infty(R;S)} \frac{\int_{-\infty}^{\infty} (u(t), g(t))_{L^2} dt}{\|g\|_{L^{q'}(R;L^{r'})}} \le C\| f\|_{L^2},$$

which proves (11.46).

This estimate describes a dispersive smoothing effect of the Schr¨odinger equation. For example, the L^r -spatial norm of the solution may blow up at some time, but it must be finite almost everywhere in t. Intuitively, this is because if the Fourier modes of the solution are sufficiently in phase at some point in space and time that they combine to form a singularity, then dispersion pulls them apart at later times.

Although the above proof of the Schrodinger equation Strichartz estimates is elementary, in the sense that given the interpolation estimate for the Schrodinger equation and the one-dimensional Hardy-Littlewood-Sobolev inequality it uses only Holder's inequality, it does not explicitly clarify the role of dispersion (beyond the dispersive decay of solutions in time). An alternative point of view is in terms of

Restriction theorems for the Fourier transform.

The Fourier solution of the Schrodinger equation (11.13) is

$$u(x,t)=\int_{R^n} f^\wedge(k)e^{ik.x+i|k|^2 t}dk.$$

Thus, the space-time Fourier transform $u(k,\tau)$ of $u(x,t)$

$$u(k,r)=\frac{1}{(2\pi)}n+1\int u(x,t)e^{ik.x+i\tau t}dxdt,$$

is a measure supported on the paraboloid $\tau+|k|^2=0$. This surface has nonsingular curvature, which is a geometrical expression of the dispersive nature of the Schrodinger equation. The Strichartz estimates describe a boundedness property of the restriction of the Fourier transform to curved surfaces.

As an illustration of this phenomenon, we state the Tomas-Stein theorem on the restriction of the Fourier transform in R^{n+1} to the unit sphere S^n.

Theorem 11.114. Suppose that $f\in L^p\left(R^{n+1}\right)$ with

$$1\le p\le\frac{2n+4}{n+4}$$

and let $g^\wedge = f^\wedge\big|_{\mathrm{S}^n}$. Then there is a constant $C(p,n)$ such that

$$\|g\|_{L^2(\mathrm{S}^n)}\le C\|f\|_{L^p(\mathbb{R}^{n+1})}.$$

11.3.2 Local L^2-solutions.

In this section, we use the Strichartz estimates for the linear Schrodinger equation to obtain a local existence result for solutions of the nonlinear Schrodinger equation with initial data in L^2.

If X is a Banach space and $T>0$, we say that $u\in C([0,T];X)$ is a mild X-valued solution of (11.43) if it satisfies the Duhamel-type integral equation

(11.49) $$u=T(t)f+i\lambda\int_0^t T(t-s)\left\{|u|^\alpha(s)u(s)\right\}ds \quad \text{for } t\in[0,T]$$

where $T(t)=e^{it\Delta}$ is the solution operator of the linear Schr¨odinger equation defined by (11.22). If a solution of (11.49) has sufficient regularity then it is also a solution of (11.43), but here we simply take

(11.49) as our definition of a solution. We suppose that $t \geq 0$ for definiteness; the same arguments apply for $t \leq 0$.

Before stating an existence theorem, we explain the idea of the proof, which is based on the contraction mapping theorem. We write (11.49) as a fixed-point equation

(11.50) $$u = \Phi(u) \qquad \Phi(u)(t) = T(t)f + i\lambda\Psi(u)(t),$$

(11.51) $$\Psi(u)(t) = \int_0^t T(t-s)\left\{|u|^{\alpha}(s)u(s)\right\}ds.$$

We want to find a Banach space E of functions $u:[0,T] \to L^r$ and a closed ball $B \subset E$ such that $\Phi : B \to B$ is a contraction mapping when $T > 0$ is sufficiently small.

As discussed in Section 11.4.3, the Schrodinger operators $T(t)$ form a strongly continuous group on L^p only if $p = 2$. Thus if $f \in L^2$, then

$$\Phi : C\left([0,T]; L^{2/(\alpha+1)}\right) \to C\left([0,T]; L^2\right),$$

but Φ does not map the space $C\left([0,T]; L^r\right)$ into itself for any exponent $1 \leq r \leq \infty$.

If α is not too large, however, there are exponents q, r such that

(11.52) $$\Phi : L^q\left(0,T; L^r\right) \to L^q\left(0,T; L^r\right).$$

This happens because, as shown by the Strichartz estimates, the linear solution operator T can regain the space-time regularity lost by the nonlinearity. (For a brief discussion of vector-valued L^p-spaces, see Section 6.A.)

To determine values of q, r for which (11.52) holds, we write

$$L^q\left(0,T; L^r\right) = L_t^q L_x^r$$

for short, and consider the action of Φ defined in (11.50)–(11.51) on such a space.

First, consider the term Tf in (11.50) which is independent of u.

Theorem 11.53

implies that $Tf \in L_t^q L_x^r$ if $f \in L^2$ for any admissible pair (q, r).

Second, consider the nonlinear term $\Psi(u)$ in (11.51). We have

$$\left\| |u|^{\alpha} u \right\| L_t^q L_x^r = \left[\int_0^T \left(\int_{\mathbb{R}^n} |u|^{r(\alpha+1)} \, dx \right)^{q/r} dt \right]^{1/q}$$

$$= \left[\int_0^T \left(\int_{\mathbb{R}^n} |u|^{r(\alpha+1)} \, dx \right)^{q(\alpha+1)/r(\alpha+1)} dt \right]^{(\alpha+1)/q(\alpha+1)}$$

$$= \|u\|^{\alpha+1}_{L_t^{r(\alpha+1)} L_x^{r(\alpha+1)}} .$$

Thus, if $u \in L_t^{q1} L_x^{r1}$ then $|u|^{\alpha} u \in L_t^{q'_2} L_x^{r'_2}$ where

(11.53) $$q_1 = q'_2 (\alpha+1), \qquad r_1 = r'_2 (\alpha+1).$$

If (q_2, r_2) is an admissible pair, then the Strichartz estimate (11.48) implies that

$$\Psi(u) \in L_t^{q_2} L_x^{r_2}.$$

In order to ensure that Ψ preserves the L_x^r-norm of u, we need to choose $r = r_1 = r_2$, which implies that $r = r'(\alpha+1)$, or

(11.54) $$r = \alpha + 2.$$

If r is given by (11.54), then it follows from Definition 11.52 that

$$(q_2, r_2) = (q, \alpha+2)$$

is an admissible pair if

(11.55) $$q = \frac{4(\alpha+2)}{n\alpha}$$

and $0 < \alpha < 4/(n-2)$, or $0 < \alpha < \infty$ if $0 < \alpha < \infty$ In that case, we have

$$\Psi : L_t^{q_1} L_x^{\alpha+2} \to L_t^q L_x^{\alpha+2}$$

Where

(11.56) $$q_1 = q'(\alpha+1).$$

In order for Ψ to map $L_t^q L_x^{\alpha+2}$ into itself, we need $L_t^{q_1} \supset L_t^q$ or $q_1 \le q$. This condition holds if $\alpha + 2 \le q \quad \alpha \le 4/n.$ In order to prove that Φ is a contraction we will interpolate in time from $L_t^{q_1}$ to L_t^q, which requires that $a_1 < q$ or $\alpha < 4/n.$

A similar existence result holds in the critical case $\alpha = 4/n$ but the proof requires a more refined argument which we do not describe here.

Thus according to this discussion,

$$\Phi : L_t^q L_x^{\alpha+2} \to L_t^q L_x^{\alpha+2}$$

if q is given by (11.55) and $0<\alpha<4/n$. This motivates the hypotheses in the following theorem.

THEOREM 11.55. Suppose that $0<\alpha<4/n$ and

$$q=\frac{4(\alpha+2)}{n\alpha}.$$

For every $f\in L^2(R^n)$, there exists

$$T=T\left(\|f\|_{L^{2,n,\alpha,\lambda}}\right)>0$$

and a unique solution u of (11.49) with

$$u\in C\left([0,T];L^2(R^n)\right)\cap L^q\left(0,T;L^{\alpha+2}(R^n)\right).$$

Moreover, the solution map $f\mapsto u$ is locally Lipschitz continuous.

PROOF. For $F>0$, let E be the Banach space

$$E=C\left([0,T];L^2\right)\cap L^q\left(0,T;L^{\alpha+2}\right)$$

with norm

(11.57) $$\|u\|_E=\max_{[0,T]}\|u(t)\|_{L^2}+\left(\int_0^T\|u(t)\|_{L^{\alpha+2}}^q\,dt\right)^{1/q}$$

and let Φ be the map in (11.50)–(11.51). We claim that $\Phi(u)$ is well-defined for $u\in E$ and $\Phi:E\to E$.

The preceding discussion shows that $\Phi(u)\in L_t^qL_x^{\alpha+2}$ if $u\in L_t^qL_x^{\alpha+2}$. Writing $C_tL_x^2=C\left([0,T];L^2\right)$, we see that $T(.)f\in C_tL_x^2$ since $f\in L^2$ and T is a strongly continuous group on L^2. Moreover, (11.47) implies that $\Psi(u)\in C_tL_x^2$ since $\Psi(u)$ is the uniform limit of smooth functions $\Psi(u_k)$ such that $u_k\to u$ in $L_t^qL_x^{\alpha+2}c.f.$

(11.71). Thus, $\Phi:E\to E$.

Next, we estimate $\|\Phi(u)\|_E$ and show that there exist positive numbers

$$T=T\left(\|f\|_{L^{2,n,\alpha,\lambda}}\right),\qquad a=a\left(\|f\|_{L^{2,n,\alpha}}\right)$$

such that Φ maps the ball

(11.58) $$B=\left\{u\in E:\|u\|_E\le a\right\}$$

into itself.

First, we estimate $\|Tf\|_E$. Since T is a unitary group, we have

(11.59) $$\|Tf\|_{C_t L_x^2} = \|f\|_{L^2}$$

while the Strichartz estimate (11.46) implies that

(11.60) $$\|Tf\|_{L_t^q L_x^{\alpha+2}} \le C\|f\|_{L^2}.$$

Thus, there is a constant C = C(n, α) such that

(11.61) $$\|Tf\|_E \le C\|f\|_{L^2}.$$

In the rest of the proof, we use C to denote a generic constant depending on n and α.

Second, we estimate $\|\Psi(u)\|_E$ where Ψ is given by (11.51). The Strichartz estimate (11.47) gives

(11.62) $$\begin{aligned}\|\Psi(u)\|_{C_t L_x^2} &\le C\left\||u|^{\alpha+1}\right\|_{L_t^{q'} L_x^{(\alpha+2)'}}\\ &\le C\|u\|^{\alpha+1}_{L_t^{q'(\alpha+1)} L_x^{(\alpha+2)'(\alpha+1)}}\\ &\le C\|u\|^{\alpha+1}_{L_t^{q'} L_x^{\alpha+2}}\end{aligned}$$

where q_1 is given by (11.56). If $\phi \in L^p(0,T)$ and $1 \le p \le q,$ then Holder's inequality

with $r = q/p \ge 1$ gives

(11.63) $$\begin{aligned}\|\phi\|_{L^p(0,T)} &= \left(\int_0^T 1.|\phi(t)|^p\, dt\right)^{1/p}\\ &\le \left[\left(\int_0^T 1^{r'}\, dt\right)^{1/r'}\left(\int_0^T |\phi(t)|^{pr}\, dt\right)^{1/r'}\right]^{1/p}\\ &\le T^{1/p-1/q}\|\phi\|_{L^q(0,T)}.\end{aligned}$$

Using this inequality with $p = q_1$ in (11.62), we get

(11.64) $$\|\Psi(u)\|_{C_t L_x^2} \le CT^\theta \|u\|^{\alpha+1}_{L_t^q L_x^{(\alpha+2)}}$$

Where $\theta = (\alpha+1)(1/q_1 - 1/q) > 0$ is given by

(11.65) $$\theta = 1 - \frac{n\alpha}{4}.$$

We estimate $\|\Psi(u)\|_{L_t^q L_x^{\alpha+2}}$ in a similar way. The Strichartz estimate (11.48) and the Holder estimate (11.63) imply that

(11.66) $$\left\|\Psi(u)\right\|_{L_t^q L_x^{\alpha+2}} \le C\|u\|_{L_t^q L_x^{\alpha+2}}^{\alpha+1} \le CT^\theta \|u\|_{L_t^q L_x^{\alpha+2}}^{\alpha+1}.$$

Thus, from (11.64) and (11.66), we have

(11.67) $$\left\|\Psi(u)\right\|_E \le CT^\theta \|u\|_{L_t^q L_x^{\alpha+2}}^{\alpha+1}.$$

Using (11.61) and (11.67), we find that there is a constant $C = C(n,\alpha)$ such that

(11.68) $$\left\|\Phi(u)\right\|_E \le \|Tf\|_E + |\lambda| \left\|\Psi(u)\right\|_E \le C\|f\|_{L^2} + C|\lambda| T^\theta \|u\|_{L_t^q L_x^{\alpha+2}}^{\alpha+1}$$

for all $u \in E$. We choose positive constants a, T such that

$$a \ge 2C\|f\|_{L^2}, \qquad 0 < 2C|\lambda| T^\theta a^\alpha \le 1.$$

Then (11.68) implies that $\Phi : B \to B$ where $B \subset E$ is the ball (11.118).

Next, we show that Φ is a contraction on B. From (11.110) we have

(11.69) $$\Phi(u) - \Phi(v) = i\lambda\left[\Psi(u) - \Psi(v)\right].$$

Using the Strichartz estimates (11.47)–(11.48) in (11.51) as before, we get

(11.70) $$\left\|\Psi(u) - \Psi(v)\right\|_E \le C\left\| |u|^\alpha u - |v|^\alpha v \right\|_{L_t^{q'} L_x^{(\alpha+2)'}}.$$

For any $\alpha > 0$ there is a constant $C(\alpha)$ such that

$$\left| |w|^\alpha w - |z|^\alpha z \right| \le C\left(|w|^\alpha + |z|^\alpha\right)|w - z| \qquad \text{for all } w, z \in \mathbb{C}.$$

Using the identity

$$(\alpha+2)' = \frac{\alpha+2}{\alpha+1}$$

and Holder's inequality with $r = \alpha+1, r' = (\alpha+1)/\alpha,$ we get that

$$\begin{aligned}
\left\| |u|^\alpha u - |v|^\alpha v \right\|_{L_x^{(\alpha+2)'}} &= \left(\int \left| |u|^\alpha u - |v|^\alpha v \right|^{(\alpha+2')} dx\right)^{1/(\alpha+2)'} \\
&\le C\left(\int \left(|u|^\alpha + |v|^\alpha\right)^{(\alpha+2)'} |u-v|^{(\alpha+2)'} dx\right)^{1/(\alpha+2)'} \\
&\le C\left(\int \left(|u|^\alpha + |v|^\alpha\right)^{r'(\alpha+2)'} dx\right)^{1/r'(\alpha+2)'} \\
&\quad \left(\int |u-v|^{r(\alpha+2)'} dx\right)^{1/r(\alpha+2)'} \\
&\le C\left(\|u\|_{L_x^{(\alpha+2)}}^\alpha + \|v\|_{L_x^{(\alpha+2)}}^\alpha\right)\|u-v\|_{L_x^{\alpha+2}}
\end{aligned}$$

We use this inequality in (11.70) followed by Holder's inequality in time to get

$$\left\|\Psi(u)-\Psi(v)\right\|_E \le C\left(\int_0^T\left[\|u\|_{L_x^{(\alpha+2)}}^{\alpha}+\|v\|_{L_x^{(\alpha+2)}}^{\alpha}\right]^{q'}\|u-v\|_{L_x^{\alpha+2}}^{q'}\,dt\right)^{1/q'}$$

$$\le C\left(\int_0^T\left[\|u\|_{L_x^{(\alpha+2)}}^{\alpha}+\|v\|_{L_x^{(\alpha+2)}}^{\alpha}\right]^{p'q'}dt\right)^{1/p'q'}$$

$$\left(\int_0^T\|u-v\|_{L_x^{\alpha+2}}^{p'q'}dt\right)^{1/pq'}.$$

Taking $p=q/q'>1$ we get

$$\left\|\Psi(u)-\Psi(v)\right\|_E \le C\left(\int_0^T\left[\|u(t)\|_{L_x^{\alpha+2}}^{\alpha p'q'}+\|v(t)\|_{L_x^{\alpha+2}}^{\alpha p'q'}\right]dt\right)^{1/pq'}\|u-v\|_{L_t^qL_x^{\alpha+2}}.$$

Interpolating in time as in (11.63), we have

$$\int_0^T\|u(t)\|_{L_x^{\alpha+2}}^{\alpha p'q'}\,dt \le \left(\int_0^T 1^{\alpha p'q'r'}dt\right)^{1/r'}\left(\int_0^T\|u(t)\|_{L_x^{\alpha+2}}^{\alpha p'q'}dt\right)^{1/r}$$

and taking $\alpha p'q'r=q,$ which implies that $1/p'q'r'=\theta$ where θ is given by (11.65),

we get

$$\left(\int_0^T\|u(t)\|_{L_x^{\alpha+2}}^{\alpha p'q'}dt\right)^{1/r}\le T^{\theta}\|u-v\|_{L_t^qL_x^{\alpha+2}}.$$

It therefore follows that

(11.71)

$$\left\|\Psi(u)-\Psi(v)\right\|_E \le CT^{\theta}\left(\|u\|_{L_t^qL_x^{\alpha+2}}^{\alpha}+\|v\|_{L_t^qL_x^{\alpha+2}}^{\alpha}\right)\|u-v\|_{L_t^qL_x^{\alpha+2}}.$$

Using this result in (11.69), we get

$$\left\|\Phi(u)-\Phi(v)\right\|_E \le C|\lambda|T^{\theta}\left(\|u\|_E^{\alpha}+\|v\|_E^{\alpha}\right)\|u-v\|_E.$$

Thus if $u,v\in B,$

$$\left\|\Phi(u)-\Phi(v)\right\|_E \le 2C|\lambda|T^{\theta}\alpha^{\alpha}\|u-v\|_E.$$

Choosing $T>0$ such that $2C|\lambda|T^{\theta}a^{\alpha}<1,$ we get that $\Phi:B\to B$ is a contraction, so it has a unique fixed point in B. Since we can choose the radius a of B as large as we wish by taking T small enough, the solution is unique in E.

The Lipshitz continuity of the solution map follows from the contraction mapping theorem. If Φ_f denotes the map in (11.110), $\Phi_{f_1},\Phi_{f_2}:B\to B$ are contractions, and u_1,u_2 are the fixed points of Φ_{f_1},Φ_{f_2}, then

$$\|u_1 - u_2\|_E \leq C\| f_1 - f_2\|_{L^2} + K\|u_1 - u_2\|_E$$

Where $K < 1$. Thus

$$\|u_1 - u_2\|_E \leq \frac{C}{1-K}\| f_1 - f_2\|_{L^2}.$$

This local existence theorem implies the global existence of L^2-solutions for subcritical nonlinearities $0 < \alpha < 4/n$ because the existence time depends only the L^2-norm of the initial data and one can show that the L^2-norm of the solution is constant in time.

For more about the extensive theory of the nonlinear Schrodinger equation and other nonlinear dispersive PDEs see, for example, [6, 29, 39, 40]. In this section, we summarize some results about Schwartz functions, tempered distributions, and the Fourier transform.

Check your progress

2. Explain about the non linear Schrodinger equation.

.---

11.4 THE SCHWARTZ SPACE

Since we will study the Fourier transform, we consider complex-valued functions.

DEFINITION 11.56. The Schwartz space $S\left(R^n\right)$ is the topological vector space of functions $f : R^n \to R$ such that $f \in C^\infty\left(R^n\right)$ and

$$x^\alpha \partial^\beta f\left(x\right) \to 0 \qquad \text{as } |x| \to \infty$$

for every pair of multi-indices $\alpha, \beta \in R_0^n$. For $\alpha, \beta \in R_0^n$ and $f \in S\left(R^n\right)$ let

(11.72) $$\| f\|_{\alpha,\beta} = \sup_{R^n}\left|x^\alpha \partial^\beta f\right|.$$

A sequence of functions $\{ f_k : k \in R\}$ converges to a function f in $S\left(R^n\right)$ if

$$\| f_n - f\|_{\alpha,\beta} \to 0 \text{ as } k \to \infty$$

for every $\alpha, \beta \in R_0^n$.

That is, the Schwartz space consists of smooth functions whose derivatives (including the function itself) decay at infinity faster than any power; we say, for short, that Schwartz functions are rapidly decreasing. When there is no ambiguity, we will write $S(R^n)$ as S.

EXAMPLE 11.57. The function $f(x) = e^{-|x|^2}$ belongs to $S(R^n)$. More generally, if p is any polynomial, then $g(x) = p(x)e^{-|x|^2}$ belongs to S.

EXAMPLE 11.58. The function

$$f(x) = \frac{1}{\left(1+|x|^2\right)^k}$$

does not belongs to S for any $k \in R$ since $|x|^{2k} f(x)$ does not decay to zero as $|x| \to \infty$.

EXAMPLE 11.59. The function $f: R \to R$ defined by

$$f(x) = e^{-x^2} \sin\left(e^{x^2}\right)$$

does not belong to $S(R)$ since $f'(x)$ does not decay to zero as $|x| \to \infty$.

The space $D(R^n)$ of smooth complex-valued functions with compact support is contained in the Schwartz space $S(R^n)$. If $f_k \to f$ in D (in the sense of Definition 3.8), then $f_k \to f$ in S, so D is continuously embedded in S. Furthermore, if $f \in S$, and $n \in C_c^\infty(R^n)$ is a cutoff function with $\eta_k(x) = \eta(x/k)$, then $\eta_k f \to f$ in S as $k \to \infty$, so D is dense in S.

The topology of S is defined by the countable family of semi-norms $\| \cdot \|_{\alpha,\beta}$ given in (11.72). This topology is not derived from a norm, but it is metrizable; for example, we can use as a metric

$$d(f,g) = \sum_{\alpha,\beta \in \mathbb{N}_0^n} \frac{C_{\alpha,\beta} \|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}}$$

where the $c_{\alpha,\beta} > 0$ are any positive constants such that $\sum_{\alpha,\beta \in \mathbb{N}_0^n} c_{\alpha,\beta}$ converges. Moreover, S is complete with respect to this metric. A complete, metrizable topological vector space whose topology may be defined by a countable family of seminorms is called a Frechet space. Thus, S is a Frechet space.

If we want to make explicit that a limit exists with respect to the Schwartz topology, we write

$$f = S - \lim_{k\to\infty} f_k,$$

and call f the S-limit of $\{f_k\}$.

If $f_k \to f$ as $k \to \infty$ in S, then $\partial^\alpha f_k \to \partial^\alpha f$ for any multi-index $\alpha \in R_0^n$. Thus, the differentiation operator $\partial^\alpha : S \to S$ is a continuous linear map on S.

11.4.1. Tempered distributions.

Tempered distributions are distributions (c.f. Section 3.3) that act continuously on Schwartz functions. Roughly speaking, we can think of tempered distributions as distributions that grow no faster than a polynomial at infinity.

DEFINITION 11.60. A tempered distribution T on R^n is a continuous linear functional $T : S\left(R^n\right) \to R.$ The topological vector space of tempered distributions

is denoted by $S'\left(R^n\right)$ or S'. If $\langle T, f\rangle$, fi denotes the value of $T \in S'$ acting on $f \in S,$ then a sequence $\{T_k\}$ converges to T in S', written $T_k \to T,$ if

$$\langle T_k, f\rangle \to \langle T, f\rangle \quad \text{fi for every } f \in S.$$

Since $D \subset S$ is densely and continuously embedded, we have $S' \subset D'$. Moreover, a distribution $T \in D'$ extends uniquely to a tempered distribution $T \in S'$ if and only if it is continuous on D with respect to the topology on S.

Every function $f \in L^1_{1oc}$ defines a regular distribution $T_f \in D'$ by

$$\langle T_f, \phi\rangle = \int f\phi dx \qquad \text{for all } \phi \in D.$$

If $|f| \le p$ is bounded by some polynomial p T, then T_f extends to a tempered distribution $T_f \in S'$, but this is not the case for functions f that grow too rapidly at infinity.

EXAMPLE 11.61. The locally integrable function $f(x)=e^{|x|^2}$ defines a regular distribution $T_f \in D''$ but this distribution does not extend to a tempered distribution.

EXAMPLE 11.62. If $f(x)=e^x\cos(e^x)$, then $T_f \in D'(R)$ extends to a tempered distribution $T \in S'(R)$ even though the values of $f(x)$ grow exponentially as $x \to \infty$.

The name 'tempered distribution' is short for 'distribution of temperate growth,' meaning polynomial growth.

This tempered distribution is the distributional derivative $T = T_g^{'}$ of the regular distribution T_g where $f = g'$ and $g(x)=\sin(e^x)$:

$$\langle f,\phi\rangle = -\langle g,\phi'\rangle - \int \sin(e^x)\phi(x)dx \quad \text{for all } \phi \in S.$$

The distribution T is decreasing in a weak sense at infinity because of the rapid oscillations of f.

EXAMPLE 11.63. The series

$$\sum_{n\in\square} \delta^{(n)(x-n)}$$

Where $\delta^{(n)}$ is the nth derivative of the δ-function converges to a distribution in $D'(R)$, but it does not converge in $S'(R)$ or define a tempered distribution.

We define the derivative of tempered distributions in the same way as for distributions. If $\alpha \in R_0^n$ is a multi-index, then

$$\langle \partial^\alpha T,\phi\rangle = (-1)^{|\alpha|}\langle T,\partial^\alpha \phi\rangle.$$

We say that a C^∞-function f is slowly growing if the function and all of its derivatives are of polynomial growth, meaning that for every $\alpha \in R_0^n$ there exists a constant C_α and an integer N_α such that

$$\left|\partial^\alpha f(x)\right| \le C_\alpha\left(1+|x|^2\right)^{N_\alpha}.$$

If f is C^∞ and slowly growing, then $f\phi \in S$ whenever $\phi \in S$, and multiplication by f is a continuous map on S. Thus for $T \in S'$, we may define the product $fT \in S'$ by

$$\langle fT,\phi\rangle = \langle T, f\phi\rangle$$

11.5 THE FOURIER TRANSFORM

The Schwartz space is a natural one to use for the Fourier transform. Differentiation and multiplication exchange rˆoles under the Fourier transformand therefore so do the properties of smoothness and rapid decrease. As a result, the Fourier transform is an automorphism of the Schwartz space. By duality, the Fourier transform is also an automorphism of the space of tempered distributions.

11.5.1. The Fourier transform on S.

DEFINITION 11.64. The Fourier transform of a function $f \in S\left(R^n\right)$ is the function $f^\wedge : R^n \to R$ defined by

(11.73)

$$f^\wedge(k) = \frac{1}{(2\pi)^n} \int f(x) e^{-ik.x} dx.$$

The inverse Fourier transform of f is the function $f^\wedge : R^n \to R$ defined by

$$f^\wedge(x) = \int f(k) e^{ik.x} dk.$$

We generally use x to denote the variable on which a function f depends and k to denote the variable on which its Fourier transform depends.

EXAMPLE 11.65. For $\sigma > 0$, the Fourier transform of the Gaussian

$$f(x) = \frac{1}{\left(2\pi\sigma^2\right)^{n/2}} e^{-|x|^2/2\sigma^2}$$

is the Gaussian

$$f^\wedge(x) = \frac{1}{(2\pi)^n} e^{-\sigma^2|k|^2/2}$$

The Fourier transform maps differentiation to multiplication by a monomial and multiplication by a monomial to differentiation. As a result, $f \in S$ if and only if $f^\wedge \in S$, and $f_n \to f$ in S if and only if $f^\wedge{}_n \to f^\wedge$ in S.

THEOREM 5.66. The Fourier transform $F: S \to S$ defined by $F: f \mapsto f^\wedge$ is a continuous, one-to-one map of S onto itself. The inverse $F^{-1}: S \to S$ is given by $F^{-1}: f \to f^\wedge$. If $f \in S$, then

$$F\left[\partial^\alpha f\right] = (ik)^\alpha f^\wedge, \qquad F\left[(-ix)^\beta f\right] = \partial^\beta f^\wedge.$$

The Fourier transform maps the convolution product of two functions to the pointwise product of their transforms.

THEOREM 5.67. If $f, g \in S,$ then the convolution $h = f * g \in S,$ and

$$h = (2\pi)^n f^\wedge g^\wedge.$$

If $f, g \in S$, then

$$\int f\bar{g}dx = (2\pi)^n \int f^\wedge g^\wedge dk.$$

In particular,

$$\int |f| dx = (2\pi)^n \int \left|f^\wedge\right|^2 dk.$$

11.5.2. The Fourier transform on S'.

The main reason to introduce tempered distributions is that their Fourier transform is also a tempered distribution.

If $\phi, \psi^\wedge \in S,$ then by Fubini's theorem

$$\int \phi\psi dx = \int \phi(x)\left[\frac{1}{(2\pi)^n}\int \psi(y)e^{-ix.y}dy\right]dx$$

$$= \int \left[\frac{1}{(2\pi)^n}\int \phi(x)e^{-ix.y}dx\right]\psi(y)dy$$

$$= \int \phi\psi dx.$$

This motivates the following definition for the Fourier transform of a tempered distribution which is compatible with the one for Schwartz functions.

DEFINITION 11.68. If $T \in S'$, then the Fourier transform $T \in S'$ is the distribution defined by

$$\left\langle \breve{T}, \phi \right\rangle = \left\langle T, \breve{\phi} \right\rangle \quad \text{for all } \phi \in S.$$

The inverse Fourier transform ˇ T ∈ S′ is the distribution defined by

$$\left\langle \breve{T}, \phi \right\rangle = \left\langle T, \breve{\phi} \right\rangle \quad \text{for all } \phi \in S.$$

We also write $\hat{T} = FT$ and $\breve{T} = F^{-1}T$. The linearity and continuity of the Fourier transform on S implies that $\hat{T}$ is a linear, continuous map on S, so the Fourier transform of a tempered distribution is a tempered distribution. The invertibility of the Fourier transform on S implies that $F : S' \to S'$ is invertible with

inverse $F^{-1} : S' \to S'$.

EXAMPLE 11.69. If δ is the delta-function supported at $0, \langle \delta, \phi \rangle = \phi(0)$, then

$$\langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0) = \frac{1}{(2\pi)^n} \int \phi(x)\, dx = \left\langle \frac{1}{(2\pi)^n}, \phi \right\rangle.$$

Thus, the Fourier transform of the δ-function is the constant function $(2\pi)^{-n}$. We may write this Fourier transform formally as

$$\delta(x) = \frac{1}{(2\pi)^n} \int e^{ik.x} dk.$$

This result is consistent with Example 11.65. We have for the Gaussian δ-sequence that

$$\frac{1}{(2\pi\sigma^2)^{n/2}} e^{-|x|^2/2\sigma^2} \to \delta \text{ in } S' \text{as } \sigma \to 0.$$

The corresponding Fourier transform of this limit is

$$\frac{1}{(2\pi)^n} e^{-\sigma^2|k|^2/2} \to \frac{1}{(2\pi)^n} \quad \text{in } S' \text{ as } \sigma \to 0.$$

If $T \in S'$, it follows directly from the definitions and the properties of Schwartz functions that

$$\langle \partial^\alpha T^\wedge, \phi \rangle = \langle \partial^\alpha T, \phi^\wedge \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi^\wedge \rangle = \left\langle T, (ik)^{\alpha\wedge} \phi \right\rangle = \left\langle T^\wedge, (ik)^\alpha \phi \right\rangle = \left\langle (ik)^\alpha T^\wedge, \phi \right\rangle,$$

with a similar result for the inverse transform. Thus,

$$\partial^\alpha T \wedge = (ik)^\alpha T^\wedge, \qquad (-ix)^\beta T^\wedge = \partial^\beta T^\wedge.$$

The Fourier transform does not define a map of the test function space D into itself, since the Fourier transform of a compactly supported function does not, in general, have compact support. Thus, the Fourier transform of a distribution $T \in D'$ is not, in general, a distribution $T^\wedge \in D'$; this explains why we define the. Fourier transform for the smaller class of tempered distributions.

The Fourier transform maps the space D onto a space Z of real-analytic functions, and one can define the Fourier transform of a general distribution $T \in D'$ as an ultradistribution $T^{\wedge} \in Z'$ acting on Z. We will not consider this theory further here.

11.5.3. The Fourier transform on L^1.

If $f \in L^1(R^n)$, then

$$\left|\int f(x) e^{-ik.x} dx\right| \le \int |f| dx,$$

so we may define the Fourier transform $f^{\wedge}$ directly by the absolutely convergent integral in (11.73). Moreover,

$$\left|f(k)\right| \le \frac{1}{(2\pi)^n} \int |f| dx.$$

It follows by approximation of f by Schwartz functions that $f^{\wedge}$ is a uniform limit of Schwartz functions, and therefore $f^{\wedge} \in C_0$ is a continuous function that approaches zero at infinity. We therefore get the following Riemann-Lebesgue lemma.

THEOREM 11.70. The Fourier transform is a bounded linear map

$$F : L^1(R^n) \to C_0(R^n) \text{ and}$$

$$\|f^{\wedge}\|_{L^\infty} \le \frac{1}{(2\pi)^n} \|f\|_{L^1}.$$

The range of the Fourier transform on L^1 is not all of C_0, however, and it is difficult to characterize.

11.5.4. The Fourier transform on L^2.

The next theorem, called Parseval's theorem, states that the Fourier transform preserves the L^2-inner product and norm, up to factors of 2π. It follows that we may extend the Fourier transform by density and continuity from S to an isomorphism on L^2 with the same properties. Explicitly, if $f \in L^2$, we choose any sequence of functions $f_k \in S$ such that f_k converges to f in L^2 as $k \to \infty$. Then we define $f^{\wedge}$ to be the L^2-limit of the $f^{\wedge}{}_k$.

Note that it is necessary to use a somewhat indirect approach to define the Fourier transform on L^2, since the Fourier integral in (11.73) does not converge if $f \in L^2 \setminus L^1$.

THEOREM 11.71. The Fourier transform $F : L^2\left(R^n\right) \to L^2\left(R^n\right)$ is a one-to-one, onto bounded linear map. If $f, g \in L^2\left(R^n\right)$, then

$$\int f\overline{g}dx = (2\pi)^n \int f^\wedge \overline{g}dk.$$

In particular,

$$\int |f|^2 dx = (2\pi)^n \int \left|f^\wedge\right|^2 dk.$$

11.5.5. The Fourier transform on L^p.

The boundedness of the Fourier transform $F : L^p \to L^{p'}$ for $1 < p < 2$ follows from its boundedness for $F : L^1 \to L^\infty$ and $F : L^2 \to L^2$ by use of the following Riesz-Thorin interpolation theorem.

THEOREM 11.72. Let Ω be a measure space and

$1 \le p_0, p_1 \le \infty, 1 \le q_0, q_1 \le \infty.$

Suppose that

$$T : L^{po}(\Omega) + L^{p1}(\Omega) \to L^{qo}(\Omega) + L^{q1}(\Omega)$$

Is a linear map such that $T : L^{pi}(\Omega) + L^{qi}(\Omega)$ for $i = 0,1$ and

$$\|Tf\|_{L^{q0}} \le M_0 \|f\|_{L^{p0}}, \qquad \|Tf\|_{L^{q1}} \le M_1 \|f\|_{L^{p1}}$$

For some constants M_0, M_1. if $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{po} + \frac{\theta}{p1}, \qquad \frac{1}{q} = \frac{1-\theta}{qo} + \frac{\theta}{q1}$$

Then $T : L^p(\Omega) \to L^q(\Omega)$ maps $L^p(\Omega)$ into $L^q(\Omega)$ and

$$\|Tf\|_{L^q} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}.$$

In this theorem, $L^{po}(\Omega) + L^{p1}(\Omega)$ denotes the vector space of all complex-valued functions of the form $f = f_0 + f_1$ where $f_0 \in L^{po}(\Omega)$ and $f_1 \in L^{p1}(\Omega)$. Note that if $qo = p_0^{'}$ and $q1 = p_1^{'}$, then $q = p'$. An immediate consequence of this theorem and the $L^1 - L^2$ estimates for the Fourier transform is the following Hausdorff-Young

theorem.

THEOREM 11.73. Suppose that $1 \le p \le 2.$ The Fourier transform is a bounded linear map $F : L^p(R^n) \to L^{p'}(R^n)$ and

$$\|Ff\|_{Lp'} \le \frac{1}{(2\pi)^n} \|f\|_{Lp}.$$

If $1 \le p < 2,$ the range of the Fourier transform on L^p is not all of $L^{p'}$, and there exist functions $f \in L^{p'}$
whose inverse Fourier transform is a tempered distribution that is not regular. Correspondingly, if $p > 2$ the range of $F : L^p \to S'$ contains non-regular distributions. For example, $1 \in L^\infty$ and $F(1) = \delta.$

Check your progress

3. Explain about Fourier transform.

.---

11.6 THE SOBOLEV SPACES $H^s(R^n)$

A function belongs to L^2 if and only if its Fourier transform belongs to L^2, and the Fourier transform preserves the L^2-norm. As a result, the Fourier transform provides a simple way to define L^2-Sobolev spaces on R^n, including ones of fractional and negative order. This approach does not generalize to L^p-Sobolev spaces with $p \ne 2,$ since there is no simple way to characterize when a function belongs to L^p in terms of its Fourier transform.

We define a function $\langle . \rangle : R^n \to R$ by

$$\langle x \rangle = \left(1 + |x|^2\right)^{\frac{1}{2}}.$$

This function grows linearly at infinity, like $|x|$, but is bounded away from zero. (There should be no confusion with the use of angular brackets to denote a duality pairing.)

DEFINITION 11.74. For $s \in R$, the Sobolev space $H^s\left(R^n\right)$ consists of all tempered distributions $f \in S'\left(R^n\right)$ whose Fourier transform $f^\wedge$ is a regular distribution such that

$$f \langle k \rangle^{2s} \left| f^\wedge(k) \right|^2 dk < \infty.$$

The inner product and norm of $f, g \in H^s$ are defined by

$$(f,g)_{H^s} = (2\pi)^n \int \langle k \rangle^{2s} f^\wedge(k) \overline{g^\wedge}(k)\, dk, \qquad \| f \|_{H^s} = (2\pi)^n \left(\int \langle k \rangle^{2s} \left| f^\wedge(k) \right|^2 dk \right)^{\frac{1}{2}}.$$

Thus, under the Fourier transform, Hs(Rn) is isomorphic to the weighted L^2 -Space

(11.74)

$$H^{\wedge s}\left(R^n\right) = \left\{ f : R^n \to R : \langle k \rangle f \in L^2 \right\},$$

with inner product

$$\left(f^\wedge, g^\wedge \right)_{H^{\wedge s}} = (2\pi)^n \int \langle k \rangle^{2s} f^\wedge \overline{g^\wedge} dk.$$

The Sobolev spaces $\left\{ H^s : s \in R \right\}$ form a decreasing scale of Hilbert spaces with H^s continuously embedded in H^r for $s > r$. If $s \in R$ is a positive integer, then $H^s\left(R^n\right)$is the usual Sobolev space of functions whose weak derivatives of order less than or equal to s belong to $L^2\left(R^n\right)$, so this notation is consistent with our previous notation.

We may give a spatial description of H^s for general $s \in R$ in terms of the pseudo-differential operator $\Lambda : S' \to S'$ with symbol $\langle k \rangle$ defined by

(11.75)

$$\Lambda = (I - \Delta)^{\frac{1}{2}}, (\Lambda f) \wedge (k) = \langle k \rangle f^\wedge(k).$$

Then $f \in H^s$ if and only if $\Lambda^s f \in L^2$, and

$$(f,g)_{H^s} = \int \left(\Lambda^s f \right) \left(\Lambda^s \overline{g} \right) dx, \qquad \| f \|_{H^s} = \left(\int \left| \Lambda^s f \right|^2 dx \right)^{\frac{1}{2}}.$$

Thus, roughly speaking, a function belongs to H^s if it has s weak derivatives (or integrals if $s < 0$) that belong to L^2.

EXAMPLE 11.75. If $\delta \in S'\left(R^n\right)$, then $\delta = (2\pi)^{-n}$ and

$$\int \langle k \rangle^{2s\delta 2} dk = \frac{1}{(2\pi)^{2n}} \int \langle k \rangle^{2s} dk$$

converges if $2s < -n$. Thus, $\delta \in H^s(R^n)$ if $s < -n/2$, which is precisely when functions in H^s are continuous and pointwise evaluation at 0 is a bounded linear functional. More generally, every compactly supported distribution belongs to H^s for some $s \in R$.

EXAMPLE 11.76. The Fourier transform of $1 \in S'$, given by $\hat{1} = \delta$, is not a regular distribution. Thus, $1 \notin H^s$ for any $s \in R$.

We let

(11.76) $$H^{\infty} = \bigcap_{s \in \mathbb{R}} H^s, \quad H^{-\infty} = \bigcup_{s \in \mathbb{R}} H^s.$$

Then $S \subset H^{\infty} \subset H^{-\infty} \subset S'$ and by the soblev embedding theorem $H^{\infty} \subset C_0^{\infty}$.

11.7 FRACTIONAL INTEGRALS

One way to approach fractional integrals and derivatives is through potential theory.

The Riesz potential. For $0 < \alpha < n$, we define the Riesz potential $I_\alpha : R^n \to R$ by

$$I_\alpha(x) = \frac{1}{\gamma_\alpha} \frac{1}{|x|^{n-\alpha}}, \quad \gamma_\alpha = \frac{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}{\Gamma(n/2 - \alpha/2)}.$$

Since $\alpha > 0$, we have $I_\alpha \in L^1_{1oc}(R^n)$

The Riesz potential of a function $\phi \in S$ is defined by

$$I_\alpha * \phi(x) = \frac{1}{\gamma\alpha} \int \frac{\phi(y)}{|x-y|^{n-\alpha}} dy.$$

The Fourier transform of this equation is

$$(I_\alpha * \phi)(k) = \frac{1}{|k|^\alpha} \phi(k).$$

Thus, we can interpret convolution with I_α as a homogeneous, spherically symmetric fractional integral operator of the order α. We write it symbolically as

$$I_\alpha * \phi = |D|^{-\alpha} \phi,$$

where $|D|$ is the operator with symbol $|k|$. In particular, if $n \geq 3$ and $\alpha = 2,$ the potential I_2 is the Green's function of the Laplacian operator,

$$-\Delta I_2 = \delta.$$

If we consider

$$|D|^{-\alpha} : L^p\left(R^n\right) \to L^q\left(R^n\right)$$

as a map from L^p to L^q, then a scaling argument similar to the one for the Sobolev embedding theorem implies that the map can be bounded only if

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}. \tag{11.77}$$

The following Hardy-Littlewood-Sobolev inequality states that this map is, in fact, bounded for $1 < p < n/\alpha$. The proof (see e.g. [18] or [27]) uses the boundedness of the Hardy-Littlewood maximal function on L^p for $1 < p < \infty$.

THEOREM 11.77. Suppose that $0 < \alpha < n, 1 < p < n/\alpha,$ and q is defined by (11.77). If $f \in L^p\left(R^n\right),$ then $I_\alpha * f \in L^q\left(R^n\right)$ and there exists a constant $C\left(n, \alpha, p\right)$ such that

$$\left\| I_\alpha * f \right\|_{L^q} \leq C \left\| f \right\|_{L^p} \text{ for every } f \in L^p\left(R^n\right).$$

This inequality may be thought of as a generalization of the Gagliardo-Nirenberg inequality in Theorem 3.28 to fractional derivatives. If $\alpha = 1,$ then $q = p*$ is the Sobolev conjugate of $p,$ and writing $f = |D| g$ we get

$$\|g\|_{L^{p^*}} \leq C \left\| \left(|D| g\right) \right\|_{L^p}.$$

The Bessel potential. The Bessel potential corresponds to the operator

$$\Lambda^{-\alpha} = \left(I - \Delta\right)^{-\alpha/2} = \left(I + |D|^2\right)^{-\alpha/2}.$$

where Λ is defined in (11.75) and $\alpha > 0$. The operator $\Lambda^{-\alpha}$ is a non-homogeneous, spherically symmetric fractional integral operator; it plays an analogous role for non-homogeneous Sobolev spaces to the fractional derivative $|D|^{-\alpha}$ for homogeneous Sobolev spaces.

If $\phi \in S,$ then

$$\left(\Lambda^{-\alpha}\phi^{\wedge}\right)(k)=\frac{1}{\left(1+|k|^{2}\right)^{\alpha/2}}\phi^{\wedge}(k).$$

Thus, by the convolution theorem,

$$\Lambda^{-\alpha}\phi = G_{\alpha} * \phi$$

where

(11.78)
$$G_{\alpha}=F^{-1}\left[\frac{1}{\left(1+|k|^{2}\right)^{\alpha/2}}\right].$$

For any $0<\alpha<\infty,$ this distributional inverse transform defines a positive function that is smooth in $R^{n}\setminus\{0\}$. For example, if $\alpha=2$, then G_{2} is the Green's function of the Helmholtz equation

$$-\Delta G_{2}+G_{2}=\delta.$$

Unlike the kernel I_{α} of the Riesz transform, however, there is no simple explicit expression for G_{α}.

For large k, the Fourier transform of the Bessel potential behaves asymptotically like the Riesz potential and the potentials have the same singular behavior at $x\rightarrow 0$. For small k, the Bessel potential behaves like $1-\left(\frac{\alpha}{2}\right)|k|^{2}$, and it decays exponentially as $|x|\rightarrow\infty$ rather than algebraically like the Riesz potential. We therefore have the following estimate.

PROPOSITION 11.78. Suppose that $0<\alpha<n$ and G_{α} is the Bessel potential defined in (11.78). Then there exists a constant $C=C(\alpha,n)$ such that

$$0<G_{\alpha}(x)\leq\frac{C}{|x|^{n-\alpha}}\ \text{ if } 0<|x|<1,\qquad 0<G_{\alpha}(x)\leq e^{\frac{-|x|}{2}}\ \text{ if } |x|\geq 1.$$

Finally, we state a version of the Sobolev embedding theorem for fractional L^{2}-Sobolev spaces.

THEOREM 11.79. If $0<s<\frac{n}{2}$ and

$$\frac{1}{q}=\frac{1}{2}-\frac{s}{n},$$

Notes

Then $H^s\left(R^n\right) \to L^q\left(R^n\right)$ and there exists a constant $C = C(n,s)$ such that

$$\|f\|_{L^q} \leq \|f\|_{H^s}.$$

If $\frac{n}{2} < s < \infty,$ then $H^s\left(R^n\right) \to C_0\left(R^n\right)$ and there exists a constant $C = C(n,s)$ such that

$$\|f\|_{L^\infty} \leq \|f\|_{H^s}.$$

Proof. The result for $s < \frac{n}{2}$ follows from Proposition 11.78 and the Hardy-Littlewood-Sobolev inequality c.f. [18].

If $s > \frac{n}{2}$, we have for $f \in S$ that

$$\begin{aligned}
\|f\|_{L^\infty} &= \sup_{x\in R^n}\left|\int f \wedge(k) e^{ik.x} dk\right| \\
&\leq \int\left| f^\wedge(k)\right| dk \\
&\leq \int \frac{1}{\left(1+|k|^2\right)^{\frac{s}{2}}}.\left(1+|k|^2\right)^{\frac{s}{2}}\left| f^\wedge(k)\right| dk \\
&\leq \left(\int \frac{1}{\left(1+|k|^2\right)^s} dk\right)^{\frac{1}{2}} \left(\int\left(1+|k|^2\right)^s \left| f^\wedge(k)\right|^2 dk\right)^{\frac{1}{2}} \\
&\leq C\|f\|_{H^s},
\end{aligned}$$

since the first integral converges when $2s > n$. Since S is dense in H^s, it follows that this inequality holds for every $f \in H^s$ and that $f \in C_0$ since f is the uniform limit of Schwartz functions.

Check your progress

4. Prove theorem 11.79

.---

11.8 LET US SUM UP

In this unit we have discussed about a Semi linear equations, the non linear Schrodinger equation, Local L^2 solutions, the Schwartz space, Tempered distributions, The fourier transforms and The Soble spaces, Fractional integrals. In order to prove a local existence result, we choose α large enough that the nonlinear term is well-behaved by Sobolev embedding, but small enough that the norm of the semigroup maps from L^2 into $H^{2\alpha}$ is integrable as $t \to 0^+$. As we will see, this is the case if $n/4 < \alpha < 1$, so we restrict attention to $1 \le n \le 3$ space dimensions. We use the Strichartz estimates for the linear Schrodinger equation to obtain a local existence result for solutions of the nonlinear Schrodinger equation with initial data in L^2.

11.9 KEY WORDS

1. Negative Laplacian on a bounded domain, with say homogeneous Dirichlet boundary conditions.
2. The Fourier representation of the semigroup operators.
3. The linear Schrodinger equation group
4. The pair of exponents (q,r) is an admissible pair if

   $\frac{2}{q} = \frac{n}{2} - \frac{n}{r}$ Where $2 < q < \infty$ and

   $2 < r < \frac{2n}{n-2}$ if $n \ge 3$ Or $2 < r < \infty$ if $n = 1, 2.$
5. A tempered distribution T on R^n is a continuous linear functional $T : S(R^n) \to R.$
6. Green's function of the Laplacian operator.

11. 10 QUESTIONS FOR REVIEW

1. Discuss about a semi linear equations.
2. Discuss about non linear Schrodinger equation.
3. Discuss about Fourier transforms.
4. Discuss about Sobolev spaces

11.11 SUGGESTED READINGS AND REFERENCES

1.S. L. Ross, Differential Equations, 3rd Edn., Wiley India, 1984.
2.DiBenedetto, Partial Differential Equations, Birkhaüser, 1995.
3.L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.
4.I.N. Sneddon Elements of Partial Differential Equations McGrawHill 1986.
5.R. Churchil & J. Brown, Fourier Series & Boundary Value Problems.
6.R.C. McOwen , Partial Differential Equations (Pearson Edu.) 2003.
7.Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum, Outline Series, McGraw hill Series.
8.Partial Differential Equations, -Walter A.Strauss
9.Partial Differential Equations,-John K.Hunter
10. Partial Differential Equations,*Erich Mieremann*
11. Partial Differential Equations,-*Victor Ivrii*

11.12 ANSWERS TO CHECK YOUR PROGRESS

1. See section 11.2
2. See section 11.3
3. See section 11.5
4. See section 11.7

UNIT-12 PARABOLIC EQUATIONS

STRUCTURE

12.0 OBJECTIVE

In this unit we will learn and understand about heat equation, general second order parabolic PDE,Definition of weak solutions, The Galerkin approximation.

12.1 INTRODUCTION

The theory of parabolic PDEs closely follows that of elliptic PDEs and, like elliptic PDEs, parabolic PDEs have strong smoothing properties.

For example, there are parabolic versions of the maximum principle and Harnack's inequality, and a Schauder theory for Holder continuous solutions .

Moreover, we may establish the existence and regularity of weak solutions of parabolic PDEs by the use of L^2-energy estimates.

12.2 THE HEAT EQUATION

Just as Laplace's equation is a prototypical example of an elliptic PDE, the heat equation

(12.1) $$u_t = \Delta u + f$$

Is a prototypical example of a parabolic PDE.

This PDE has to be supplemented by suitable initial and boundary conditions to give a well-posed problem with a unique solution.

As an example of such a problem, consider the following IBVP with Direchlet BCs on a bounded open set $\Omega \subset R^n$ for $u : \Omega \times [0, \infty) \to R$:

$$u_t = \Delta u + f(x,t) \qquad for\ x \in \Omega\ and\ t > 0,$$

(12.2) $$u(x,t) = 0$$

$$for\ x \in \partial\Omega\ and\ t > 0,$$

$$u(x,0) = g(x) \qquad for\ x \in \Omega.$$

Here $f : \Omega \times (0, \infty) \to R$ and $g : \Omega \to R$ are a given forcing term and initial condition.

This problem describes the evolution in time of the temperature $u(x,y)$ of a body occupying the region Ω containing a heat source f per unit volume, whose boundary is held at fixed zero temperature and whose initial temperature is g.

One important estimate $(in\, L^\infty)$ for solutions of (12.2) follows from the maximum principle. If $f \leq 0$, corresponding to 'heat' sinks', then for any $T > 0$,

$$\max_{\bar{\Omega}\times[0,T]} u \leq \max\left[0, \max_{\bar{\Omega}} g\right]$$

To derive this inequality, note that if u is a smooth function which attains a maximum at $x \in \Omega$ and $0 < t \leq T$, then $u_t = 0$ if $0 < t < T$ or $u_t \geq 0$ $t = T$ and $\Delta u \leq 0$. Thus $u_t - \Delta u \geq 0$ which is impossible if $f < 0$, so u attains its maximum on $\partial\Omega \times [0, T]$, where $u = 0$, or at t = 0. The result for $f \leq 0$ follows by a perturbation argument. The physical interpretation of this maximum principle in terms of thermal diffusion is that a local "hotspot" cannot develop spontaneously in the interior when no heat sources are present. Similarly, if $f \geq 0$, we have the minimum principle

$$\min_{\bar{\Omega}\times[0,T]} u \geq \min\left[0, \min_{\bar{\Omega}} g \quad\right]$$

Another basic estimate for the heat equation $\left(in\, L^2\right)$ follows from an integration of the equation. We multiply (12.1) by u, integrate over Ω, apply the divergence theorem, and use the BC that $u = 0$ on $\partial\Omega$ to obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |Du|^2 dx = \int_\Omega fu\, dx\,.$$

Integrating this equation with respect to time and using the initial condition, we get

(12.3)

$$\frac{1}{2}\int_\Omega u^2\left(x,t\right)dx + \int_0^t\int_\Omega |Du|^2 dxds = \int_0^t\int_\Omega fu\, dx\, ds + \frac{1}{2}\int_\Omega g^2 dx\,.$$

For $0 \leq t \leq T$, we have from the Cauchy inequality with $\in$ that

$$\int_0^t\int_\Omega fu\, dx\, ds \leq \left(\int_0^t\int_\Omega f^2\, dx\, ds\right)^{1/2}\left(\int_0^t\int_\Omega u^2 dxds\right)^{1/2}$$

$$\leq \frac{1}{4\in}\int_0^T\int_\Omega f^2 dxds + \in\int_0^T\int_\Omega u^2\, dx\, ds$$

$$\leq \frac{1}{4\in}\int_0^T\int_\Omega f^2 dxds + \in T\max_{0\leq t\leq T}\int_\Omega u^2\, dx\,.$$

Notes

Thus, taking the supremum of (12.3) over $t \in [0,T]$ and using this inequality with $\in T = 1/4$ in the result, we get

$$\frac{1}{4}\max_{[0,T]}\int_{\Omega} u^2(x,t)\,dx + \int_0^T\int_{\Omega}|Du|^2\,dxdt \le T\int_0^T\int_{\Omega} f^2\,dxdt + \frac{1}{2}\int_{\Omega} g^2\,dx$$

.

It follows that we have an a priori energy estimate of the form

(12.4)

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H_0^1)}\,dx\,dt \le T\int_0^T\int_{\Omega} f^2 dxdt + \frac{1}{2}\int_{\Omega} g^2 dx\,.$$

Where $C = C(T)$ is a constant depending only on T. We will use this energy estimate to construct weak solutions. The parabolic smoothing of the heat equation is evident from the fact that if $f = 0$, say we can estimate not only the solution u but its derivative Du in terms of the initial data g.

Check your progress

1. Explain about semi linear heat equation.

12.3 GENERAL SECOND-ORDER PARABOLIC PDES

The qualitative properties of (12.1) are almost unchanged if we replace the Laplacian $-\Delta$ by any uniformly elliptic operator L on $\Omega \times (0,T)$. We write L in divergence form as

(12.5) $$L = -\sum_{i,j=1}^{n} \partial_i\left(a^{ij}\,\partial_j u\right) + \sum_{j=1}^{n} b^j \partial_j u + cu$$

Where $a^{ij}(x,t), b^{i}(x,t), c(x,t)$ are coefficient function with $a^{ij} = a^{ji}$. We assume that these exists $\theta > 0$ such that

(12.12) $\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$ for all $(x,t) \in \Omega \times (0,T)$ and $\xi \in R^n$.

In fact, we will use a slightly better estimate in which $\|f\|_{L^2(0,T;L^2)}$ is replaced by the weaker norm $\|f\|_{L^2(0,T;H^{-1})}$.

The corresponding parabolic PDE is then

(12.7) $u_t + \sum_{j=1}^{n} b^j \partial_j u + cu = \sum_{i,j=1}^{n} \left(a^{ij}\partial_j u\right) + f$.

Equation (12.7) describes evolution of a temperature field u under the combined effect of diffusion a^{ij}, advection b^i, linear growth or decay c, and external heat sources f.

The corresponding IBVP with homogeneous Dirichlet BCs is

$$u_t + Lu = f,$$

(12.8) $$u(x,t) = 0 \quad for\ x \in \partial\Omega\ and\ t > 0$$

$$u(x,0) = g(x) \qquad for\ x \in \bar{\Omega}.$$

Essentially the same estimates hold for this problem as for the heat equation. To begin with, we use the L^2-energy estimates to prove the existence of suitably defined weak solutions of (12.8).

Check your progress

2.Explain about general second order parabolic PDE's.

12.4 DEFINITION OF WEAK SOLUTIONS

To formulate a definition of a weak solution of (12.8), we first suppose that the domain Ω, the coefficients of L, and the solution u are smooth. Multiplying (12.7), by a test function $v \in C_c^\infty(\Omega)$, integrating the result over Ω, and applying the divergence theorem, we get

(12.9) $\left(u_t(t),v\right)_{L^2} + a\left(u(t),v;t\right) = f\left((t),v\right)_{L^2}$ for $0 \le t \le T$

Where $(.,.)_{L^2}$ denotes the L^2-inner product

$$(u,v)_{L^2} = \int_\Omega u(x)v(x)\,dx,$$

And a is the bilinear form associated with L

$$a(u,v;t) = \sum_{i,j=1}^{n} \int_\Omega a^{ij}(x,t)\partial_i u(x)\partial_j u(x)dx$$

(12.10) $+\sum_{j=1}^{n} \int_\Omega b^j(x,t)\partial_j u(x)v(x)dx + \int_\Omega c(x,t)u(x)v(x)\,dx$

In (12.9), we have switched to the "vector-valued", and write $u(t) = u(.,t)$.

To define weak solutions, we generalize (12.9) in a natural way. In order to ensure that the definition makes sence, we make the following assumptions.

ASSUMPTION 12.1: The set $\Omega \subset R^n$ is bounded and open, $T > 0$, and ;

(1) the coefficient of a in (12.10) satisfy $a^{ij}, b^j, c \in L^\infty\left(\Omega \times (0,T)\right)$;

(2) $a^{ij} = a^{ji}$ for $1 \le i,j \le n$ and the uniform elliptic condition (12.6) holds for some constant $\theta > 0$;

(3) $F \in L^2\left(0,T;H^{-1}(\Omega)\right)$ and $g \in L^2(\Omega)$.

Here, we allow f to take values in $H^{-1}(\Omega) = H_0^1(\Omega)'$. We denote the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ by

$$\langle .,. \rangle : H^{-1}(\Omega) \times H_0^1(\Omega) \to R$$

Parabolic Equations

Since the coefficients of *a* are uniformly bounded in time, it follows , $a : H_0^1(\Omega) \times H_0^1(\Omega) \times (0,T) \to R$.

Moreover, there exist constants C > 0 and $\gamma \in R$ such that for every $u, v \in H_0^1(\Omega)$

(12.11) $$C\|u\|_{H_0^1}^2 \leq a(u,u;t) + \gamma\|u\|_{L^2}^2,$$

(12.12) $$|a(u,v;t)| \leq C\|u\|_{H_0^1}\|v\|_{H_0^1}.$$

We then define weak solutions (12.8) as follows.

Definition 12.2. A function $u : [0,T] \to H_0^1(\Omega)$ is a weak solution of (12.8) if:

(1) $u \in L^2(0,T; H_0^1(\Omega))$ *and* $u_1 \in L^2(0,T; H^{-1}(\Omega))$;

(2) For every $v \in H_0^1(\Omega)$,

(12.13) $$\langle u_t(t), v \rangle + a(u(t), v; t) = \langle f(t), v \rangle$$

For t pointwise a.e. in [0,T] where *a* is defined in (12.10)

(3) $u(0) = g$.

The PDE is imposed in a weak sense by (12.13) and the boundary condition $u = 0$ on $\partial\Omega$ by the requirement that $u(t) \in H_0^1(\Omega)$. Two points about this definition deserve comment.

First, the time derivative u_t in (12.13) is understood as a distributional time derivative; that is $u_t = w$ if

(12.14) $$\int_0^T \phi(t)u(t)\,dt = -\int_0^T \phi'(t)w(t)\,dt$$

For every $\phi:(0,T)\to R$ with $\phi\in C_c^\infty(0,T)$. This is a direct generalization of the notion of the weak derivative of a real-valued function. The integrals in (12.14) are vector-valued Lebesgue integrals (Bochner integrals). Which ar defined in an analogous way to the Lebesgue integral of an integrable real-valued function as the L^1-limit of integrals of simple functions. See section 12.A for further discussion of such integrals and the weak derivative of vector-valued functions. Equation (12.13) may then be understood in a distributional sense as an equation of the weak derivative u_t on (0, T).

Second, it is not immediately obvious that the initial condition $u(0)=g$ in Definition 12.2 makes sense. We do not explicitly require any continuity on u, and since $u\in L^2\left(0,T;H_0^1(\Omega)\right)$ is defined only up to point wise every where equivalence in $t\in[0,T]$ it is not clear that specifying a point wise value $t=0$ imposes any restriction on u. As shown in Theorem 12.41, however, the conditions that $u\in L^2\left(0,T;H_0^1(\Omega)\right)$ and $u_t\in L^2\left(0,T;H^{-1}(\Omega)\right)$ imply that $u\in C\left([0,T];L^2(\Omega)\right)$. Therefore, identifying u with its continuous representative, we see that the initial condition makes sense.

We then have the following existence result, whose proof will be given in the following sections.

THEOREM 12.3. Suppose that the conditions in Assumption 12.1 are satisfied. Then for every $f\in L^2\left(0,T;H^{-1}(\Omega)\right)$ and $g\in H_0^1(\Omega)$ there is a unique weak solution

$u\in C\left([0,T];L^2(\Omega)\right)\cap L^2\left(0,T;H_0^1(\Omega)\right)$ of (12.8), in the sense of Definition 12.2, with $u_t\in L^2\left(0,T;H^{-1}(\Omega)\right)$. Moreover, there is a constant C, depending only on Ω, T, and the coefficients of L, such that

$$\|u\|_{L^\infty\left(0,T;L^2\right)}+\|u\|_{L^2\left(0,T;H_0^1\right)}+\|u_t\|_{L^2\left(0,T;H^{-1}\right)}\le C\left(\|f\|_{L^2\left(0,T;H^{-1}\right)}+\|g\|_{L^2}\right)$$

12.5 THE GALERKIN APPROXIMATION

The basic idea of the existence proof is to approximate $u:[0,T]\to H_0^1(\Omega)$ by function $u_N:[0,T]\to E_N$ that take values in a finite-dimensional subspace $E_N\subset H_0^1(\Omega)$ of dimension N. To obtain the u_N, we project the PDE only E_N, meaning that we require that u_N satisfies the PDE up to a residual which is orthogonal to E_N. This gives a system of ODEs for u_N, which has a solution by standard ODE theory. Each u_N satisfies an energy estimate of the same form as the a priori estimate for solutions of the PDE.These estimates are uniform in N, which allows us to pass to the limit $N\to\infty$ and obtain a solution of the PDE.

In more detail, the existence of uniform bunds implies that the sequence $\{u_N\}$ is weakly compact in a suitable space and hence, by the Banach-Alaoglu theorem, there is a weakly convergent subsequence $\{u_{N_k}\}$ such that $u_{N_k}\to u$ as $k\to\infty$. Since the PDE and the approximating ODEs are linear, and linear functionals are continuous with respect to weak convergence, the weak limit of the solutions of the ODE is a solutions of the PDE. As with any similar compactness argument, we get existence but not uniqueness, since it is conceivable that different subsequence of approximate solutions could converge to different weak solutions. We can, however, prove uniqueness of a weak solution directly from the energy estimates. Once we have weak convergence $u_N\to u$ of the full approximate sequence. One can than prove that the sequence, in fact, converges strongly in $L^2(0,T;H_0^1)$.

Method such as this one, in which we approximate the solution of a PDE by the projection of the solution and the equation into finite dimensional subspaces, are called Galerkin methods. Such methods have close connections with the variational formulation of PDEs. For example, in the time-independent case of an elliptic PDE given by a variational principle, we may approximate the minimization problem

Notes

over a finite-dimensional subspace E_N. The corresponding equations for a critical point are a finite-dimensional approximation of the weak formulation of the original PDE. We may then show, under suitable assumptions, that as $N \to \infty$ solutions u_N of finite-dimensional minimization problem approach a solution *u* of the original problem.

There is considerable flexibility the finite-dimensional spaces E_N one useds in a Galerkin method. For our analysis, we take

(12.15) $$E_N = \langle w_1, w_2, \ldots..w_N \rangle$$

To be the linear space spanned by the first N vectors in an orthonormal basis $\{w_k : k \in N\}$ of $L^2(\Omega)$, which we may also assume to be an orthogonal basis of $H_0^1(\Omega)$. For definiteness, take the $w_k(x)$ to be the eigenfunctions of the Dirichlet Laplacian on Ω:

(12.16) $$-\Delta_{w_k} = \lambda_k w_k \qquad w_k \in H_0^1(\Omega) \quad \mathit{for} \quad k \in N$$

From the previous existence theory for solutions of elliptic. PDEs, the Dirichlet Laplacian on a bounded open set is a self-adjoint operator with compact resolving, so that suitably normalized set of eigenfunctions have the required properties.

Explicitly, we have

$$\int_\Omega w_j w_k dx = \{1 if \ j = k; 0 if \ j \neq k\}$$
$$\int_\Omega Dw_j Dw_k dx = \{\lambda j if \ j = k; 0 if \ j \neq k\}$$

We may expand any $u \in L^2(\Omega)$ in L^2-convergent series as

$$u(x) = \sum_{k \in N} c^k w_k(x)$$

Where $c_k = (u, w_k)_{L^2}$ and $u \in L^2(\Omega)$ if and only if

$$\sum_{k \in N} |c^k|^2 < \infty$$

Similarly, $u \in H_0^1(\Omega)$, and the series converges in $H_0^1(\Omega)$. If and only if

$$\sum_{k \in N} \lambda_k \left| c^k \right|^2 < \infty$$

We denote by P_N the orthogonal projections:

$$P_N : H_0^1(\Omega) \to E_N \subset H_0^1(\Omega) \text{ or}$$

$$P_N : H_0^{-1}(\Omega) \to E_N \subset H_0^{-1}(\Omega)$$

Which we obtain by restricting or extending P_N from $L^2(\Omega)\, to\, H_0^1(\Omega)\, or\, H_0^{-1}(\Omega)$ respectively.

Thus P_N is defined on $H_0^1(\Omega)$ by 12.17 and on $H_0^{-1}(\Omega)$ by

$$\left(P_N u, v\right) = (u, P_N v) \; for\ all\, v \in H_0^1(\Omega)$$

While this choice of E_N is convenient for our existence proof, other choices are useful in different contexts.

For example, the finite-element method is a numerical implementation of the Galerkin method which uses a space E_N of piecewise polynomial functions that are supported on simplices, or some other kind of element.

Unlike the eigenfunctions of the Laplacian, finite-element basis functions, which are supported on a small number of adjacent elements, are straightforward to construct explicitly.

Furthermore, one can approximate functions on domains with complicated geometry in terms of the finite-element basis functions by subdividing the domain into simplices, and one can refine the decomposition in regions where higher resolution is required.

The finite-element basis functions are not exactly orthogonal, but they are almost orthogonal since they overlap only if they are sup- ported on nearby elements.

As a result, the associated Galerkin equations involve sparse matrices, which is crucial for their efficient numerical solution.

One can obtain rigorous convergence proofs for finite-element methods that are similar to the proof discussed here.

12.6 UNIQUENESSS OF WEAK SOLUTIONS

If u_1, u_2 are two solutions with the same data f,g, then by linearity $u = u_1 - u_2$ is a solution with zero data $f = 0, g = 0$. To show uniqueness, it is therefore

sufficient to show that the only weak solution with zero data is u=0.

Since $u(t) \in H_0^1(\Omega),$ we may take $\upsilon = u(t)$ as a test function in (6.13), with f=0, to get

$$\langle u_t, u \rangle + a(u, u; t) = 0,$$

12.7 PARABOLIC EQUATIONS

Where this equation holds pointwise a.e. in $[O, T]$ in the sense of weak derivatives. Using the coercitvity estimate 12.11. we find that there are constants $\beta > 0 \, and - \infty < \gamma < \infty$ such that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^{2} + \beta\|u\|_{H_0^1}^{2} \leq \gamma\|u\|_{L^2}^{2}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^{2} + \gamma\|u\|_{L^2}^{2}, u(0) = 0,$$

And since $\|u(0)\| L^2 = 0,$ Grownwall’s inequality implies that

$$\|u(t)\| L^2 = 0 \text{ for all } t \geq 0, so\, u = 0.$$

In a similar way, we get continuous dependence of weak solutions on the data. If u_i is the weak solution with data $f_i, g_i \, for \, i = 1, 2,$ then there is a constant C independent of the data such that

$$\|u_1 - u_2\| L^{\infty}(0, T, L^2) + \|u_1 - u_2\| L^{2(O,T;H_0^1})$$

$$\leq C\left(\| f_1 - f_2\| L^2(O,T;H^{-1}) + \|g_1 - g_2\| L^2\right)$$

12.5.5. Regularity of weak solutions, For operators with smooth coefficients on smooth domains with smooth data F,g, one can obtain regularity results for weak solutions by deriving energy estimates for higher-order derivatives of the approximate Galerkin solutions u_N and taking the limit as $N \to \infty.$ A repeated application of this procedure, and the Sobolev theorem, implies, from the Sobolev embedding theorem, that the weak solutions constructed above are smooth, classifcal solutions if the data satisfy appropriate compatibility relations.

12.8 A SEMILINEAR HEAT EQUATION

The Galerkin method is not restricted to linear or scalar equations. In this section, we briefly discuss it application to a Semilinear heat equation. For more information and examples of the application Galerkin methods to nonlinear evolutionary PDEs.

Parabolic IBVP for u(x,t)

$$u_t = \Delta u - f(u) \quad in \, \Omega \times (O, T),$$

(12,28) u=0 $\quad on \, \partial\Omega \times (O, T),$

$$u(x, 0) = g(x) \quad on \, \Omega \times \{0\},$$

We suppose, for simplicity, that

(12,29) $\quad f(u) \sum_{k=0}^{2p-1} c_k u^k$

Is a polynomial of odd degree $2p-1 \geq 1.$ We also assume that the coefficient $c2p-1>0$ of the highest degree term is positive. We then have the following global existence result

Theorem 12.8 Let T>0. For every $g \in L^2(\Omega),$ there is a unique weak solution

$$u \in C\left([0,T]; L^2(\Omega) \cap L^2(O,T; H_0^1(\Omega)\right) \cap L^{2p}(0,T; L^{2p}(\Omega)).$$

Of (12,28)-(12.29).

The proof follows the standard Galerkin method for a parabolic PDE. We will not give it in detail, but we comment on the main new difficulty that arises as a result of the nonlinearity.

Toi obtain the basic a priori energy estimate, we multiplying the PDE by u,

$$\left(\frac{1}{2}u^2\right)_t + |Du|^2 + uf(u) = div(uDu),$$

And integrate the result over Ω, using the divergence theorem and the boundary

Condition, which gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|Du\|_{L^2}^2 + \int_\Omega uf(u)dx = 0.$$

Since $uf(u)$ is an every polynomial of degree 2p with positive leading order coefficient and the measure $|\Omega|$ is finite, there are constants $A>0, C\geq 0$ such that

$$A\|u\| L_{2p}^{2p} \leq \int_\Omega uf(u)dx + C.$$

We therefore have that

$$(12.30)\quad \frac{1}{2}\sup_{[O,T]}\|u\|_{L^2}^2 + \int_0^T \|Du\|_{L^2}^2\, dt + A\int_0^T \|u\|_{2p}^{2P} dt \leq CT + \frac{1}{2}\|g\|_{L^2}^2.$$

Note that if $\|u\| L^{2p}$ is finite then $\|u\| L^{2p}$ is finite for $q=(2p)^{'}$, , since then $q(2p-1)=2p\, and$

$$\int_{\Omega} \left| f(u)^q dx \le A \right| \int_{\Omega} |u|^{q(2p-1)} dx + C \le A \|u\| L^{2p} + C.$$

Thus, in giving a weak formulation of PDE, we want to use test functions

$$v \in H_0^1(\Omega) \cap L^{2p}(\Omega)$$

So that both $(Du, Dv)_{L^2}$ *and* $(f(u), v)_{L^2}$ are well-defined.

The Galerkin approximations $\{u_N\}$ take values in a finite dimensional subspace $E_N \subset H_0^1(\Omega) \cap L^{2p}(\Omega)$ and satisfy

$$uN_t = \Delta u_N + P_N$$

Where PN is the orthogonal projection onto $E_N in L^2(\Omega)$ .These approximations satisfy the same estimates as the a priori estimates in (12.30). The Galerkin ODEs have a unique local solution since the nonlinear terms are Lipschitz continuous functions of u_N. Moreover, in view of the a priori estimates, the local solutions remain bounded, and therefore they exist globally for $0 \le t < \infty$.

Since the estimates hold uniformly in N, we extract a subsequence that converges weakly (or weak-star) $u_N \to u$ in the appropriate topologies to a limiting function

$$u \in L^{\infty}(0,T;L^2) \cap L^2(O,T;H_0^1) \cap L^{2p}(O,T;L^{2p}).$$

Moreover, from the equation

$$u_t \in L^2(0,T;H^{-1}) + L^q(O,T;H^{-1}) + L^q(O,T;L^q)$$

Where $q = (2p)^{'}$ is the Holder conjugate of 2p.

In order to prove that u is a solution PDE, however, we have to show that

$$f(u_N) \to f(u)$$

PARABOLIC EQUATIONS

In an appropriate sense. This is not immediately clear because of the lack of weak

Continuity of nonlinear functions; in general, even if $f(u_N) \to \bar{f}$

converges, we may not have $\bar{f} = f(u)$. To show (12)), we use the compactness Theorem 12.9 stated below. This theorem and the weak convergence properties found above imply that there is a subsequence of approximate solutions such that

$$u_N \to u \text{ strongly in } L^2(0,T;L^2).$$

This is equivalent to strong L^2 convergence on $\Omega \times (0,T)$. By the Riesz-Fischer theorem, we can therefore extract a subsequence so that $u_N(x,t) \to u(x,t)$ pointwise a.e. on $\Omega \times (0,T)$.

Using the dominated convergence theorem and the uniform bounds on the approximate solutions, we find that for every $v \in H_0^1(\Omega) \cap L^{2p}(\Omega)$

$$(f(u_N(t)), \upsilon_{L^2} \to (f(u(t)), v)_{L^2}$$

Pointwise, a.e. on $[0,T]$.

Finally, we state the compactness theorem used here

THEOREM 12.9. Suppose that $X \to Y \to Z$ are branch spaces, where X , Z are reflective and X is compactly embedded in Y 1 Let $< p < \infty.$ If the functions $(0,T) \to X$ are such that $\{u_N\}$ is uniformly bounded in $L^p(0,T;Z),$ and $\{u_{Nt}\}$ is uniformly bounded in $u_N \to u$ and is uniformly bounded in $L^p(0,T;Z),$ then there is a subsequence that converges strongly in $L^2(0,T;Y).$

The proof of this theorem is based on Ehrling's lemma.

LEMMA 12.10. Suppose that $X \to Y \to Z$ are Banach spaces, where X is compactly embedded in Y. For any $\in > 0$ there exists a constant $C_\in$ such that

$$\|u_n\|_Y > \in \|u_n\|_X + n\|u_n\|_Z$$

PROOF. If not, there exists $n \in$ N. since $\{u_N\}$ is bounded in X and X is complicity embedded in Y, there is a sequence, which we still donated by $\{u_N\}$ that converges strongly in Y, to u, say. Then $\{\|u_n\| y\}$ is bounded and therefore $u = 0$ from (12.00). However, (12.00) also implies that $\|u_n\| y > c$ for every $n \in R,$ which is a contradiction.

If we do not impose a sign condition on the nonlinearity, then solutions may 'blow up' in finite time, as for the ODE $u_t = u^3,$ and then we do not get global existence.

EXAMPLE 12.11. Consider the following one-dimensional IBVP for $u(x,t)$ in $0 < x < 1, t > 0:$

$$u_t = u_{xx} + u^3.$$

(6.33)

$$u(0,t) = u = (1,t) = 0.$$
$$u(x,0)g(x).$$

Suppose that $u(x,t)$ is smooth solution and let

$$c(t) = \int_0^1 u(x,t)\sin(\pi, x)dx$$

A SEMILINEAR HEAT EQUATION

Denote the first Fourier coefficient of u . Multiplying the PDE by $\sin(\pi, x)$, integrating with respect to x over (0,1), and using Green's formula to write

$$\int_0^1 u_{xx}(x,t)\sin(\pi, x)dx = [u_x \sin(\pi, x) - ru\cos(\pi)x]_0^1 - \pi^2 \int_0^1 u(x,t)\sin(\pi, x)dx = -\pi^2 c.$$

$$u_N \to u$$

We get that

$$\frac{dc}{dt} = -\pi^2 c + \int_0^1 u^3 \sin(\pi, x)dx.$$

Notes

Now suppose that $g(x) \geq 0$. Then the maximum principle implies that $u(x,t) \geq 0$ for all $0 < x < 1, t > 0$. It then follows from Holder inequality that

$$\int_0^1 u \sin(\pi, x) dx = \int_0^1 [u^3 \sin(\pi, x)^{2/3} dx$$

$$\leq \left(\int_0^1 u^3 \sin(\pi, x) dx \right)^{1/3} \left(\int_0^1 \sin(\pi, x) dx \right)^{2/3}$$

$$\leq \left(\frac{2}{\pi} \right)^{2/3} \left(\int_0^1 u^3 \sin(\pi, x) dx \right)^{1/3}$$

Hence

$$\int_0^1 u^3 \sin(\pi, x) dx \geq \frac{\pi^2}{4} c^3.$$

and therefore

$$\frac{dc}{dt} \geq \pi^2 \left(-c + \frac{1}{4} c^3 \right).$$

Thus, if c(0) > 2, Gronwall's inequality implies that

$$c(t) \geq y(t)$$

Where y(t) is the solution of the ODE

$$\frac{dy}{dt} = \pi^2 \left(-y + \frac{1}{4} y^3 \right)$$

This solution is given explicitly by

$$y(t) = \frac{2}{\sqrt{1 - e^{2\pi^2 (t - t.)}}}$$

This solution approaches infinity as $t \to t_.^-$ where, with $y(0) = c(0)$,

$$t_. = \frac{1}{\pi^2} \log \frac{c(0)}{\sqrt{c(0)^2 - 4}}$$

Therefore no smooth solution of (12.100) can exist beyond $t = t.$.

The argument used in the previos example does not prove that c(t) blows up at $t = t.$ It is conceivable that the solution loses smoothness at an earlier time – for example, because another Fourier coefficient blows up first – thereby invalidating the argument that c(t) blows up. We only get a sharp result if the quantity proven to blows up is a 'controlling norm', meaning that local smooth solutions exist so long as the controlling norm,' meaning that local smooth solutions exist so long as the controlling norm remains finite.

EXAMPLE 12.12. Beale-Kato-Majda (1984) proved that solutions of the incompressible euler equations from fluid mechanics in three-space dimensions remain smooth unless

$$\int_0^t \|x(s)\| L \times (r^3)^{ds \to \infty} as\, t \to t_{.}^{-}$$

Where denotes the vorticity (the curl of the fluid velocity u(x,t)).

Thus, the $L^1(0,t;L^\infty(\mathbb{R}^3:R^3)))$ norm of ω is a controlling norm for the three dimentional incompressible Euler equations. It is open question whether or not this norm can blow up in finite time

12.9 THE NAVIER-STOKES EQUATION

Leray (1934) used a galerkin method to prove the global existence of weak solutions of the incompressible Navier-Stokes equations. In the case of three space dimensions, Leray's result has not been essentially improved upon since then, and the smoothness and uniqueness of these weak solutions remains an open question. We briefly describe leray's result here and indicate the main ideas of its proof. For a detailed discussions, see e.g. .

The incompressible Navier-Stokes equations for the velocity $u(x,t) \in R^n$ and pressure $p(x,t) \in R$ of a viscous fluid flowing in n space dimensions. Where n=2,3. And subject to an external body force $f(x,t) \in R^n$ is the following nonlinear system of PDEs:

$$u_{it} + \sum_{j=1}^{n} u_j \partial_j u_i + \partial_i p = v \sum_{j=1}^{n} \partial_j \partial_j u_i + f_{i.}$$

Notes

$$\sum_{j=1}^{n} \partial_j u_j = 0.$$

The analysis described here is based on treating the Navier-Stokes equations as a nonlinear perturbation of the linear parabolic Stokes equations

$$u_t + u.\nabla u + \nabla p = f,$$

(6.35) div u=0.

These equations apply to low –Reynolds number (high non dimensionalized viscosity) flows, which is the typical regime for large-scale flows (e.g. airplanes or oceans). The nonlinearity of the Euler equations makes them difficult to analyze, especially in three space dimensions.5 Moreover, the higher-order viscous terms $v\nabla u$ in the Navier-Stokes equation is a singular perturbation of the Euler equations, and the limiting behavior of the Navier-Stokes equations as $v \to 0$ is a subtle issue.

Check your progress

1. Prove: Suppose that $X \to Y \to Z$ are Banach spaces, where X is compactly embedded in Y. For any $\in > 0$ there exists a constant $C_{\in}$ such that

$$\|u_n\|_Y > \in \|u_n\|_X + n\|u_n\|_Z$$

12.10 LET US SUM UP

In this unit we have discussed about heat equation, general second order parabolic PDE, Definition of weak solutions, the Galerkin approximation.

$$\int_0^T (uNt, \phi\omega)_{L^2} dt = \langle\langle uNt, \phi\omega \rangle\rangle \to \langle\langle Nt, \phi\omega \rangle\rangle = \int_0^T \langle ut, \phi\omega \rangle dt.$$

Moreover, the boundedness of a in 6.12 implies similarly that

$$\int_0^T a(uN(t),\phi(t)w;t)_{L^2}dt \to \int_0^T a(u(t)\phi w;t)dt.$$

It therefore follows that u satisfies

(6.26) $\int_0^T \phi[\langle u_t, w\rangle + a(u,w;t)]dt = \int_0^T \phi\langle f, w\rangle dt.$

Since this holds for every $\phi \in C_c^\infty(O,T)$, we have

(6.27) $\langle u_t, w\rangle + a(u,w;t) = \langle f, w\rangle$

Point wise a,e, in (O,T) for every $w \in E_M$. Moreover ,since

$\bigcup_{M\in N} E_M$

Is dense in H_0^1, this equation holds for every $w \in H_0^1$, and therefore u satisfies (6.13).

Finally, to sow that the limit satisfies the initial condition $u(0) = g$, we use the integration by parts formula. Theorem (6.42) with $\in C^\infty([0,T])$ such that $\phi(0) = 1$ and $\phi(T) = 0$ to get

$$\int_0^T \langle u_t, \phi w\rangle dt = \langle u(o), w\rangle - \int_0^T \phi t\langle u, w\rangle.$$

Thus, using (6.27), we have

$$\langle u(o), w\rangle = \int_0^T \phi_t\langle u, w\rangle + \int_0^T \phi[\langle f, w\rangle - a(u,w;t)]dt.$$

Similarly, for Galerkin approximation with $w \in E_M and\ N \geq M, we\ get$

$$\langle g, w\rangle = \int_0^T \phi_t\langle u_N, w\rangle + \int_0^T \phi[\langle f, w\rangle - a(u_N,w;t)]dt.$$

Taking the limit of this equation as $N \to \infty$, when the right-hand side converges to the right-hand side of the prevous eqation , we find that $\langle u(o), w\rangle = \langle g, w\rangle$ for every $w \in E_M$, which implies that $u(0) = g$.

the heat equation (12.1) $u_t = \Delta u + f$

The heat equation $u_t = \Delta u + f$. For every

$f \in L^2(0,T;H^{-1}(\Omega))$ and $g \in H_0^1(\Omega)$ there is a unique weak solution. Galerkin method which uses a space E_N of piecewise polynomial functions that are supported on simples, or some other kind of element.

Unlike the eigenfunctions of the Laplacian, finite-element basis functions, which are supported on a small number of adjacent elements, are straightforward to construct explicitly.

12.11 KEYWORDS

1.The heat equation is $u_t = \Delta u + f$

2.For every $f \in L^2(0,T;H^{-1}(\Omega))$ and $g \in H_0^1(\Omega)$ there is a unique weak solution .

3.The basic idea of the existence proof is to approximate $u:[0,T] \to H_0^1(\Omega)$ by function $u_N:[0,T] \to E_N$ that take values in a finite-dimensional subspace $E_N \subset H_0^1(\Omega)$ of dimension N.

4.Parabolic PDE has to be supplemented by suitable initial and boundary conditions to give a well-posed problem with a unique solution.

12.12 QUESTIONS FOR REVIEW

1. Discuss about general second order Parabolic PDE
2. Discuss about weak solutions
3. Discuss about Galekin approximation

12.13 SUGGESTED READINGS AND REFERENCES

1. S. L. Ross, Differential Equations, 3rd Edn., Wiley India, 1984.

2. DiBenedetto, Partial Differential Equations, Birkhaüser, 1995.

3. L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.

4. I.N. Sneddon Elements of Partial Differential Equations McGrawHill 1986.

5. R. Churchil & J. Brown, Fourier Series & Boundary Value Problems.

6. R.C. McOwen , Partial Differential Equations (Pearson Edu.) 2003.

7. Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum, Outline Series, McGraw hill Series.

8. Partial Differential Equations, -Walter A.Strauss

9. Partial Differential Equations,-John K.Hunter

10. Partial Differential Equations,*Erich Mieremann*

11. Partial Differential Equations,-*Victor Ivrii*

12.14 ANSWERS TO CHECK YOUR PROGRESS

1. See section 12.4
2. See section 12.9
3. See section 12.9

UNIT-13 SOLUTION OF WAVE EQUATION

STRUTURE

13.0 OBJECTIVE

By this end of this chapter we will learn and understand about Solution of 1-D wave equation, Kirchoff's formula, Solution of 2-D wave equation, Solution of wave equation for $n \geq 3$, Solution for odd n, Solution for even n, Solution of Non-homogeneous wave equation and energy methods.

13.1 INTRODUCTION

In this lesson, we seek the solution of wave equation. The homogeneous wave equation

$$u_{tt} - \Delta u = 0$$

Where $t > 0, x \in U \subset R^n$ is open and

$$u: \overline{U} \times [0,\infty) \to R.$$

The non-homogeneous wave equation.

$$u_{tt} - \Delta u = f(n,t)$$

Where $f : U \times [0,\infty) \to R$ is a prescribed function.

13.2 SOLUTION OF 1-D WAVE EQUATION

First we find the solution of wave equation in the one dimensional case. Consider the initial value problem

$$u_{tt} - u_{xx} = 0 \quad in \quad R \times (0,\infty) \qquad (1)$$

$$u = g \qquad u_t = h \quad on\ R \times \{t = 0\} \qquad (2)$$

where g and h are prescribed functions. Factorizing equation (1)

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0 \qquad (3)$$

Let

$$\upsilon(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t) \qquad (4)$$

From (3) and (4)

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \upsilon\,(x,t) = 0 \qquad x \in R, t > 0$$

υ_t+υ_x=0 (5)

which is a transport equation with constant coefficients whose solution is

$$\upsilon\,(x,t) = a(x - t) \qquad (6)$$

Where

$$a(x) := \upsilon(x,0) \qquad (7)$$

Notes

Using equation (6) in equation (4)

$$u_t - u_x = a(x-t) \ \ in\ R \times (0,\infty)$$

$$= f(x,t)$$

This is a non-homogeneous transport equation whose solution is

$$u(x,t) = \int_0^t f\,(x+t-s,s)ds + b(x+t)$$

Where

$$b(x) = u(x,0)$$

or

$$u(x,t) = \int_0^t f\,(x+t-2s)ds + b(x+t)$$

Changing the variable $x + t - 2s = y$

$$u(x,t) = \frac{1}{2}\int_{\mathrm{x-t}}^{2+t} a\,(\mathrm{y})\mathrm{dy} + b(x+t)$$

Using equation (2)

$$g\,(x) = b(x\,)$$

so $\quad u(x,t) = g(x+t) + \frac{1}{2}\int_{\mathrm{x-t}}^{x+t} a\,(y)dy$

To find a (x): We have

$$u_t\,(x,0) - u_x\,(x,0) = \upsilon\,(x,0)$$

$$h(x) - g^{\wedge\prime}\,(x) = a(x) \qquad\qquad (using\ (2)$$

$$u(x,t) = g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} \lfloor h(y) - g'(\mathrm{y})\rfloor dy$$

or

$$u(x,t) = \frac{1}{2}[g(x+t) - g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \qquad x\epsilon R, t > 0$$

(8)

This is required solution of wave equation. Equation (8) is known as D-Alembert's formula.

Note. The general solution of 1-D wave equation

$(u_t + u_x)(u_t - u_x) = 0$

Is the sum of general solution of $u_t + u_x = 0 \quad and \quad u_t - u_x = 0$

i.e. $u(x,t) = F\ (x+t) + G(x-t)$

To find the solution of wave equation over $R^n (n \geq 2)$, we first prove a

Lemma.

Def. We define

$$U\ (x; r, t) = \oint_{\partial B(x,r)} u(y,t) ds(y)$$

$$G(x; r, t) = \oint_{\partial B(x,r)} g(y)\ ds\ (y)$$

$$H\ (x; r, t) = \oint_{\partial B(x,r)} h(y)\ ds\ (y)$$

Lemma. Fix x∈R^n,satisfying

$$u_{tt} - \Delta u = 0\ in\ R^n \times (0, \infty) \tag{1}$$

$$u = g, u_t = h \quad \text{on } R^n \times \{t = 0\} \tag{2}$$

then

$U \in C^m(\overline{R_+} \times [0, \infty))\ and$

$$u_{tt} - u_{rr} - \tfrac{n-1}{r} U_r = 0 \quad in\ R_+ \times (0, \infty) \tag{3}$$

$$U = G, \ \ U_t = H \quad on\ R_+ \times \{t = 0\} \tag{4}$$

Equation (3) is known as Euler Poisson Darboux Equation.

Proof. We know

$$U(x; r, t) = \oint_{\partial B(x,r)} u\ (y,t) ds(y)$$

Shifting to unit Ball B (0,1)

$$U(x;r,t) \quad = \oint_{\partial B(0,1)} u(x+rz)ds(z)$$

Differentiating w.r.t. r

$$U_r = \oint_{\partial B(0,1)} Du(x+rz).z\, ds(z)$$

$$= \oint_{\partial B(x,r)} Du(y).\frac{y-x}{r} ds\,(y)$$

$$= \oint_{\partial B(x,r)} Du(y).v\; ds\,(y)$$

(where v is unit outward normal).

$$= \oint_{\partial B(x,r)} \frac{\partial u}{\partial v} ds\,(y)$$

$$= \frac{1}{n\propto(n)r^{n-1}} \oint_{\partial B(x,r)} \frac{\partial u}{\partial v}\; ds(y)$$

$$= \frac{1}{n\propto(x)r^{n-1}} \oint_{B(x,r)} \Delta u\;\; dy \qquad (By\; Green's\;\; formula\,)$$

$$\frac{r}{n} \oint_{B(x,r)} \Delta u\, dy$$

Hence

$$U_r(x;r,t) = \frac{r}{n} \oint_{B(x,r)} \Delta u\;\; dy$$

(5)

Again differentiating w.r.t. r

$$U_{rr}(x;r,t) = \frac{1}{n\propto(n)} \frac{\partial}{\partial r} \left[\frac{1}{r^{n-1}} \oint_{B(x,r)} \Delta u \quad dy \right]$$

$$= \frac{1-n}{n \propto (n) r^n} \oint_{B(x,r)} \Delta u \ dy + \frac{1}{n \propto (n) r^{n-1}} \oint_{\partial B(x,r)} \Delta u \, ds$$

$$\left(\frac{1}{n} - 1\right) \oint_{B(2,r)} \Delta u \ dy$$

$$+ \oint_{B(x,r)} \Delta u \ ds \qquad (6)$$

From equation (5) and (6), we observe that

$$\operatorname*{lt}_{r \to 0} U_r \ (x;, r, t) = 0$$

$$\operatorname*{lt}_{r \to 0} U_{rr} \ = \Delta u + \left(\frac{1}{n} - 1\right) \Delta u$$

$$= \frac{1}{2} \Delta u(x, \text{扨})$$

So

$$U \in C^m \overline{(R_+} \times [0, \infty))$$

By equation (5)

$$U_r(x;, r, t) = \frac{r}{n} \oint_{B(x,r)} \Delta u \, dy$$

$$= \frac{r}{n} \oint_{B(x,r)} u_{tt} \, dy$$

$$= \frac{1}{n \propto (n) r^{n-1}} \oint_{B(x,r)} u_{tt} \ dy$$

$$r^{n-1} U_r \ =: \frac{1}{n \propto (n)} \oint_{B(x,r)} u_{tt} \ dy$$

Differentiating w.r.t. r

$$r^{n-1}U_{rr} + (n+1)r^{n-2}U_r =: \frac{1}{n \propto (n)} \frac{d}{dr}\left[\oint_{B(x,r)} u_{tt}\ dy\right]$$

$$= \frac{1}{n \propto (n)} \oint_{\partial B(x,r)} u_{tt}\, ds$$

$$r^{n-1} \oint_{\partial B(x,r)} u_{tt}\ ds$$

or

$$U_{rr} + \tfrac{(n-1)}{r}U_r = U_{t\mathbb{W}} \qquad (7)$$

Which is required equation.

Also $u = g\ on\ R^n \times \{t = 0\}$

$$\oint_{\partial B(x,r)} u(y,0)\ dS(y) = \oint_{\partial B(x,r)} g(y)\ dS(y)$$

Dividing by $n \propto (n)r^{n-1}$

$U(x,0) = G(x)$

Similarly we can show

$U_t(x,0) = H(x) \quad for\ R_+ \times \{t = 0\}$

Check Your Progress

1. Explain about D-Alembert's formula.

13.3 KIRCHOFF'S FORMULA

Consider the initial value problem

$$u_{tt} - \Delta u = 0 \qquad in\ R^3 \times (0,\infty)$$

$u(x) = g(x), u_t = h \quad on\ R^3 \times \{t = 0\}$

Sol. First we prove that

$\widetilde{U}_{tt} - \widetilde{U}_{rr} = 0\ ,n\ R_+ \times (0,\infty)$

$\widetilde{U} - \tilde{G}\ ,\widetilde{U}_t = \widetilde{H} \quad on\ R_+ \times \{t = 0\}$

$\widetilde{U} = 0 \qquad on\ \{r = 0\} \times (0,\infty)$

(4)

Where $\widetilde{U} := rU \qquad \tilde{G} := rG \quad \widetilde{H} := rH$

We know Euler Poisson Darboux Equation for n=3 is

$U_{tt} - U_{rr} - \frac{2}{r}\ U_r = 0 \quad in\ R_+ \times (0,\infty)$

(5)

$U = G, \quad U_t = H \quad on\ R_+ \times \{t = 0\}$

(6)

$\overline{U}_{tt} := rU_{tt}$

$= r\{U_{rr} + \frac{2}{r}U_r\} \qquad (using\ \ 5)$

$= rU_{rr} + 2rU_r$

$= (rU_r +\ U)_r$

$= (\widetilde{U}_r)_r$

$= \widetilde{U}_{rr}$ (7)

So $\widetilde{U}$ satisfies the 1-D wave equation.

Also $\widetilde{U}$(r,0) = rU(r,0)

=rG(r)

=$\tilde{G}$

Similarly $\widetilde{U}_t(r,0) =\ \widetilde{H}(r)$

Hence, by D Alembert's formula, we have $0 \le r \le t$ r

$\widetilde{U}\ \ (x,r;t) = \frac{1}{2}\left[\tilde{G}\ (t+r) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} \widetilde{H}\ (y)\ dy$

(8)

Now

$u\ (x,t) = \lim_{r\to 0} U(x,r:t) \qquad (by\ def.)$

$$\underset{r\to 0}{\text{lt}} \frac{\widetilde{U}(x, r; t)}{r}$$

$$= \underset{r\to 0}{\text{lt}} \left\{ \left[\frac{\tilde{G}\,(t+r) - \tilde{G}\,(t+r)}{2r} \right] + \frac{1}{2r} \int_{t-r}^{t+r} \widetilde{H}\,(y)\,dy \right\}$$

$$= \tilde{G}'(t) + \widetilde{H}\,(t)$$

$$= \frac{\partial}{\partial t}\left[t \oint_{\partial B(x,t)} g\,ds \right] + t \oint_{22(x,t)} h\;\, ds \qquad (9)$$

But

$$= \frac{\partial}{\partial t}\left[t \oint_{\partial B(x,t)} g ds \right] == \frac{\partial}{\partial t}\left[\oint_{\partial B(0,1)} g(x+tz) ds(z) \right]$$

$$= \oint_{\partial B(0,1)} Dg(x + tz0.z\; ds(z)$$

$$= \oint_{\partial B(x,t)} Dg(y).\left(\frac{y-x}{t}\right)\, ds\,(y)$$

so

$$u(x,t) = \oint_{\partial B(x,t)} g ds + \oint_{\partial B(x,t)} Dg(y)(y-x)ds(y)$$

$$+ \oint_{\partial B(x,t)} t\; h\,(y)ds(y)$$

$$= \oint_{\partial B(x,t)} [g + t\; h(y) + D\; g(y)(y-x)ds(y) \qquad 10)$$

This is required solution. Equation (10) is known as kirchoff's formula.

Check Your Progress

2. Explain about Kirchoff's formula.

13.4 SOLUTION OF 2-D WAVE EQUATION

Now we find the solution of wave equation by the method of descent.

Consider initial value problem

$u_{tt} - \Delta u = 0 \quad in\ R^2 \times (0, \infty)$

$u = g\ , \quad u_t = h \quad on\ \ R^2 \times \{t = 0\}$

Sol. We regard it as a problem for *n* = 3 in which the third spatial

variable x_3 does not appear. Let us write

$u(x_1, x_{2,}, x_{3,}t) :=\ u(x_1, x_{2,}t)$

So equation (1) and (2) are modified to

$\bar{u}_{tt} - \Delta\bar{u} = 0\ \ in\ \ R^3 \times (0, \infty)$

$\bar{u}\ = g, \bar{u}_t = \bar{h}\ \ on\ R^3\ \times \{t = 0\}$

Where

$g(x_1, x_{2,}, x_{3,}) :=\ g(x_1, x_{2,})$

$\bar{h}(x_1, x_{2,}, x_{3,}) :=\ g(x_1, x_{2,})$

$If\ x = (x_1, x_2) \in R^2\ then\ \bar{x}\ \in R^3$

The solution of initial value problem defined in equation (4) and (5) is

Given by kirchoff's formula i.e.

$$\bar{u}(\bar{x}, t) = \frac{\partial}{\partial t}\left[t \oint_{\partial\overline{B}(\overline{x},t)} \overline{g}\ \overline{ds}\right] +\ t \oint_{\partial\overline{B}(\overline{x},t)} \overline{h}\ \overline{ds}$$

Where $\partial\overline{B}(\overline{x}, t)$ denotes the ball in R^3 with centre $\overline{x}$ and radius t>0 and *d* $\overline{s}$ denotes the two- dimensional surface measure on $\partial\overline{B}(\overline{x}, t)$.

Now

$$\oint_{\partial\overline{B}(\overline{x},t)} \overline{g} \;\; d\,\overline{\mathrm{S}} = \frac{1}{4\pi t^2} \int_{\partial\overline{B}(\overline{x},t)} \overline{g}\, d\overline{S}$$

$$= \frac{2}{4\pi t^2} \int_{B(x,t)} g(y)\left[1+\left(\frac{\partial\gamma}{\partial y}\right)^2\right]^{1/2} dy$$

Where factor '2' is taken as $B(\overline{x},t)$ consists of two hemisphere and

$\gamma(y) = \sqrt{t^2 - |y-x|^2}$ is the parametric equation of any $y \in B(x,t)$

$$\oint_{\partial\overline{B}(\overline{x},t)} \overline{g} \;\; d\overline{\mathrm{S}} = \frac{1}{2\pi t^2} \int_{B(x,t)} g(y)\frac{t}{\sqrt{t^2-|y-x|^2}}dy$$

$$= \frac{t}{2}\left[\frac{1}{\pi t^2} \int_{B(x,t)} g(y)\frac{t}{\sqrt{t^2-|2-x|^2}}dy\right]$$

$$= \frac{t}{2} \oint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy$$

Similarly,

$$\int_{B(x,t)} h\; dS = \frac{t}{2} \oint_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}dy$$

Using (7) and (8) in equation (6)

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy\,)\right)$$

$$+\frac{t^2}{2} \oint_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}dy$$

$$\frac{\partial}{\partial t}\left(t^2 \oint_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy\right) = \frac{\partial}{\partial t}\left(t \oint_{B(0,1)} \frac{g(x+tz)}{\sqrt{1-|z|^2}}dz\right)$$

$$= \oint_{B(x,t)} \frac{g(x+tz)}{\sqrt{1-|z|^2}}2z + t \oint_{B(0,1)} \frac{D.g(x+tz)^z}{\sqrt{1-|z|^2}}dz$$

$$= t \oint_{B(x,t)} \frac{g(y)dy}{\sqrt{t^2-|y-x|^2}} + t \oint_{B(x,t)} \frac{Dg(y).(y-x)dy}{\sqrt{t^2-|y-x|^2}}$$

Hence equation (9) gives

$$u(x,t) = \frac{t}{2} \oint_{B(x,t)} \frac{g(y)dy}{\sqrt{t^2-|y-x|^2}} + \frac{t}{2} \oint_{B(x,t)} \frac{Dg(y).(y-x)dy}{\sqrt{t^2-|y-x|^2}}$$

$$= \frac{t^2}{2} \oint_{B(x,t)} \frac{h(y)dy}{\sqrt{t^2-|y-x|^2}}$$

$$u(x,t) = \frac{1}{2} \oint_{B(x,t)} \frac{t\, g(y)+t^2 h(y)+Dy(y).(y-x)}{\sqrt{t^2-|y-x|^2}} dy \text{ ---------------(10)}$$

Where $x \in R^2$.Eq. (10) is required solution Equation (10) is known as

Poisson's Formula.

Check your progress

3. Explain about Poisson's formula.

13.5 SOLUTION OF WAVE EQUATION FOR $n \geq 3$

To find the solution of wave equation for n>3 we derive some identities.

Suppose $\emptyset : R \to R\ be\ C^{k+1}\ for\ k = 1,2,$ ....

I. $\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\emptyset(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\emptyset}{dr}\right)$

II. $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\emptyset(r)\right) = \sum_{j=0}^{k-1} \beta_j^k\, r^{j+1}\, \frac{d^j\emptyset(r)}{dr^j}$

Where

$$\beta_j^k (j = 0,1,2, \dots, K-1) are\ independe2t\ of\ r.$$

III. $\beta_0^k = 1.3.5 \dots .(2k-1)$

Proof

Notes

I. We prove it by induction. For *k=1*. We have to show

$$\frac{d^2}{dr^2}\left(r\emptyset(r)\right) = \left(\frac{1}{r}\ \frac{d}{dr}\right)\left(r^2\ \frac{d\emptyset}{dr}\right)$$

$$= \frac{d}{dr}\left[r\emptyset'^{(r)} + \emptyset(r)\right]$$

$$= r\emptyset"(r) + 2\emptyset'(r)$$

$$= \frac{1}{r}[r^2\emptyset"(r) + 2r\emptyset'(r)]$$

$$= \frac{1}{r}\frac{d}{dr}[r^2\emptyset'(r)] = \mathrm{R.H.S.}$$

Suppose result holds for *k*. So

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\emptyset(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\emptyset}{dr}\right)$$

We have to prove for *k+1* i.e

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k+1}\emptyset(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k+1}\left(r^{2k+2}\frac{d\emptyset}{dr}\right)$$

Now

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k+1}\emptyset(r)\right)$$

$$= \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left[\frac{1}{r}\frac{d}{dr}\{r^{2k+1}\emptyset(r)\}\right]$$

$$= \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left[\frac{1}{r}(2k+1)r^{2k}\emptyset(?) + r^{2k}\emptyset'^{(r)}\right]$$

$$= \frac{d^2}{dD^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left[(2k+1)r^{2k-1}\emptyset(r) + r^{2k-1}\{\emptyset'^{(r)}\}\right]\ \left(Using\ (a)\right)$$

$$= (2k+1)\left(\frac{1}{r}\frac{d}{dr}\right)^{k}[r^{2k}\emptyset'(r)] + \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left[r^{2k}\frac{d}{dr}\{r\emptyset'\}\right]$$

$$= (2k+1)\left(\frac{1}{r}\frac{d}{?r}\right)^{k}[r^{2k}\emptyset'(r)] + \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left[r^{2k}\emptyset' + r^{2k+1}\emptyset"\right]$$

$$= \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left[(2k+2)r^{2k}\emptyset' + r^{2k+1}\emptyset"(r)\right]$$

$$= \left(\frac{1}{r}\frac{d}{dr}\right)^k \frac{1}{r}\left[(2k+2)r^{2k+1}\emptyset' + r^{2k+2}\emptyset''(r)\right]$$

$$= \left(\frac{1}{r}\frac{d}{dr}\right)^k \frac{1}{r}\ \frac{d}{dr}\ [r^{2k+2}\emptyset']$$

$$= \left(\frac{1}{r}\frac{d}{dr}\right)^{k+1}\ [r^{2k+2}\emptyset']$$

Hence the result holds for k + 1.

So result is true for all k = 1,2,….

Def. Assume n is odd ,say n = $2k+1\ ,(k \geq 1)$. we define

$$\widetilde{U}(r,t) \coloneqq \left(\frac{1}{r}\frac{d}{\square r}\right)^{k-1}\left(r^{2k-1}U(x;r,t)\right)$$

$$\widetilde{G}(r,t) \coloneqq \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}G(x;r)\right)$$

$$\widetilde{H}(r,t) \coloneqq \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}H(x;r)\right)$$

$$\widetilde{U}(r,0) = \widetilde{G}(r), \widetilde{U}_t(r,0) = \widetilde{H}(r)$$

Lemma. $\widetilde{U}$ satisfies the 1-D wave equation.

$$\widetilde{U}_{t\rangle} - \widetilde{U}_{rr} = 0 \qquad in \quad R_+ \times (0,\infty)$$

$$\widetilde{U} = \widetilde{G}; \quad \widetilde{U}_t = \widetilde{H} \qquad on \quad R_+ \times \{t=0\}$$

$$\widetilde{U} = 0\ on\ \{r=0\} \times (0,\infty)$$

Proof. $U_{rr} = \left(\frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\ \frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}U)$

$$= \left(\frac{1}{r}\ \frac{\partial}{\partial r}\right)^k \left(r^{2\langle}\frac{\partial U}{\partial r}\right) \quad (by\ identity\ I)$$

$$= \left(\frac{1}{r}\ \frac{\partial}{\partial r}\right)^{k-1}\left[\frac{1}{r}\ \frac{\partial}{\partial r}\left(r^{2k}\frac{\partial U}{\partial r}\right)\right]$$

$$= \left(\frac{1}{r}\ \frac{\partial}{\partial r}\right)^{k-1}\left[\frac{2k}{r}\ r^{2k-1}U_r + r^{2k-1}U_{rr}\right]$$

$$= \left(\frac{1}{r}\ \frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}U_{rr} + 2K\ r^{2\square-2}U_r\right]$$

$$= \left(\frac{1}{r}\ \frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}\{U_{rr} + \frac{(n-1)}{r}U_r\}\right]$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} [\, r^{2k-1} U_{tt}]$$

$$= \widetilde{U}_{tt}$$

Also

$$\widetilde{U} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (\, r^{2k-1} U)$$

$$\sum_j \beta_j^k r^{j+1} \frac{d^j U}{dr^j} \qquad (\textbf{by identity II})$$

$\widetilde{U}(0,t) = 0.$

By definition $\widetilde{U}(r,0) = \widetilde{\mathrm{G}}$

$$\widetilde{U}(r,0) = \widetilde{H} \qquad on\ R_+ \times \{t = 0\}$$

Hence the lemma.

13.5.1. Solution for odd n$(n \geq 3)$

Consider the initial value problem

$$u_{tt} - \Delta u = 0 \quad in \quad R^n \times (0, \infty) \qquad (1)$$

$$u = g \quad , \quad u_t = h \quad on\ R^n \times \{t = 0\} \qquad (2)$$

Solution. By lemma, $\widetilde{U}$ satisfies the 1 – D wave equation and the initial condition. Therefore, by D Alembert's formula, on half-line $0 \leq r < t$

$$\widetilde{U}(r,t) = \frac{1}{2}\left[\tilde{G}(t+r) - \tilde{G}(t-r)\right]$$

$$+ \frac{1}{2}\int_{t-r}^{t+r} \widetilde{H}\,(y)\,dy \qquad (3)$$

for all $r \in R, t \geq 0$

$$\widetilde{U}(r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left(\, r^{2k-1} U(x;, r, t)\right)$$

$$= \beta_0^k\, rU + \beta_1^k\, r^2 \frac{\partial U}{\partial r} + \cdots$$

$\Rightarrow \frac{\widetilde{U}(r,t)}{\beta_0^k r} = U + O(r)$

Taking limit as $r \to 0$

$$\underset{r\to 0}{\text{lt}} \frac{\widetilde{U}(r,t)}{\beta_0^k r} = \underset{r\to 0}{\text{lt}} U(x,r;t) = \underset{r\to 0}{\text{lt}} \oint_{\partial B(x,r)} u(y)dS\,(y)$$

$$= 2\,(x,t)$$

So

$$u(x,t) = \frac{1}{\beta_0^k} \underset{r\to 0}{\text{lt}} \left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r} \int_{t-r}^{t+r} \widetilde{H}(y)dy\right.$$

$$= \frac{1}{\beta_0^k} \left[\tilde{G}'(t) + \widetilde{H}'(t)\right]$$

Since $n = 2k + 1$

$\beta_0^k = 1.3 \ldots .2k - 1$

$= 1.3 \ldots .(n-2) \qquad\qquad (\because n = 2K + 1)$

$= \gamma_n\,(say)$

Hence,

$$u(x,t) = \frac{1}{\gamma_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2} \oint_{\partial B(x,t)} g ds\right)\right.$$

$$\left. + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2} \oint_{\partial B(x,t)} h\ ds\right)\right]$$

Is required solution for odd n.

Note. Putting $n = 3$, we obtain kirschoff's formula.

13.5.2 Solution for even n

Suppose that n is even i.e. $n \geq 2.2m = n + 2\,, so\ m \geq 2$.

We again use the method of Descent.

Consider the initial value problem

$u_{tt} - \Delta u\ = 0\ \ in\ \ \ R^n \times (0,\infty) \qquad (1)$

$u = g \quad , \quad u_t = h\ \ on\ R^n \times \{t = 0) \qquad (2)$

Sol. Since n is even, n+1 is odd.

Notes

Suppose

$\bar{u}(x_1, x_{2,\ldots,} x_{n+1}, t) := u\,(x_1, x_{2,\ldots,} x_n, t)$

(3)

is the solution of wave equation in $R^{n+1} \times (0, \infty) i.e.$

$\bar{u}_{tt} - \Delta\bar{u} = 0 \quad in \quad R^{n+1} \times (0, \infty)$

(4)

With initial condition.

$\bar{u} = \bar{g} \quad , \quad \bar{u}_t = \bar{h} \quad on\ R^{n+1} \times \{t = 0\}$

(5)

Where

$\bar{g}(x_1, x_{2,\ldots,} x_{n+1}) := g\,(x_1, x_{2,\ldots,} x_n)$

$\bar{h}(x_1, x_{2,\ldots,} x_{n+1}) := h\,(x_1, x_{2,\ldots,} x_n)$

The solution of equation (4) subject to (5) is

$$\bar{u}(\bar{x}, t) = \frac{1}{\gamma_{n+1}} \left\{ \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(t^{n+1} \oint_{\partial\bar{B}(\bar{x}, \langle)} \bar{g}\, d\bar{s} \right) + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(t^{n-1} \oint_{\partial\bar{B}(\bar{x}, t)} \bar{h}\, d\bar{s} \right) \right\}$$

Where $\bar{B}(\bar{x}, t)$ denotes the ball in R^{n+1} with centre $\bar{x}$ and radius t and

d$\bar{s}$ denotes the n-dimensional surface measure on $\partial\bar{B}(\bar{x}, t)$

Now

$$\oint_{\partial\bar{B}(\square, t)} \bar{g}\, d\bar{s} = \frac{2}{(n+1) \propto (n+1) t^n} \int_{B(x,t)} g(y)[1 + |D\gamma(y)|^2]^{-1/2}\, dy$$

Where the factor ‘2 ‘is due to the fact that the surface area consists of

two hemispheres and $\partial\bar{B}(\bar{x}, t) \cap (y_{n+1} \geq 0)$ has the equation

$\gamma(y) = \sqrt{t^2 - |y - x|^2}\,, y \in B(x, t) and\ \partial\bar{B}(\bar{x}, t) \cap (y_{n+1} \leq 0)$ is the graph of $-\gamma(y)$.

$$\oint_{\partial\bar{B}(\bar{x},t)} \bar{g}d\bar{s} = \frac{2}{(n+1) \propto (n+1)t^n} \int_{B(x,t)} g(y)\left[\frac{1}{\sqrt{t^2-|y-x|^2}}\right]dy$$

$$= \frac{2 \propto (n)t}{(n+1) \propto (n-1)} \oint_{\partial B(x,t)} \frac{g(y)d2}{\sqrt{t^2-|y-x|^2}}$$

Similarly

$$\oint_{\partial\bar{B}(\bar{x},t)} \bar{h}d\bar{s} = \frac{2}{(n+1) \propto (n+1)} \int_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}dy$$

Using equation (7) and (8) in equation (6)

$$u(x,t)$$

$$= \frac{1}{\gamma_{n+1}} \frac{\propto (n)}{(n+1) \propto (n+1)}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{2-2}{2}}\left(t^n \oint_{B(x,t)} \frac{gdy}{\sqrt{t^2-|y-x|^2}}\right)\right.$$
$$\left.+ \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \oint_{B(x,t)} \frac{h\,dy}{t^2-|y-x|^2}\int \right)\right]$$

But

$$\frac{\propto (n)}{\gamma_{n+1}(n+1) \propto (n+1)} = \frac{1}{2.1 \ldots .(n-2) \langle}$$

$$= \frac{1}{\gamma_n}\ (say)$$

Hence

$$u(x,t) = \frac{1}{\gamma_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \oint_{B(x,t)} \frac{gdy}{\sqrt{t^2-|y-x|^2}}\right)\right.$$
$$\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \oint_{B(x,t)} \frac{h\,dy}{\sqrt{t^2-|y-x|^2}}\right)\right]$$

Is required solution, where n is even.

Note. For $n = 2, \gamma_n = 2, we\ get\ the\ poisson'sformula.$

Consider the initial value problem

$$u_{tt} - \Delta u = f \quad in \quad R^n \times (0, \infty) \qquad (1)$$

$$u = 0 \quad , \quad u_t = 0 \quad on\ R^n \times \{t = 0\} \qquad (2)$$

Where $f \in C^{[n/2]+1}\left(R^n \times (0, \infty)\right)$; $[n/2]$ denotes the greatest integer function, then solution of equation (1) subject to (2) is

$$u(x,t) = \int_0^t u(x,t;s)ds \qquad x \in R^n\ t \geq 0 \qquad (3)$$

Where $u(x,t;s)$is a solution of

$u_{tt}(x,t;s) - \Delta u\,(x,t;s) = 0 \quad in \quad R^n \times (s, \infty)$

$u(x,t;s) = 0;\ \ u_t(x,t;s) = f(x,t;s)on\ R^n \times \{t = s\}$

(4)

Sol. To show that equation (3) is a solution of equation (1) subject to (2) we need to show

(i) $u \in C^2(R^n \times [0, \infty)$

(ii) $u_{tt} - \Delta u = f(x,t) \ \ in \quad R^n \times (0, \infty)$

(iii) $\underset{(x,t)\to(x^0,0)}{lt} \quad u(x,t) = 0$

$\underset{(x,t) \to (x^0, 0)}{lt} u_t(x, P) = 0$

For each point $x^0 \in R^n$.

Proof. (i) $\left[\frac{n}{2}\right]$ denotes the greatest integer function.

If n is even $\left[\frac{n}{2}\right]$+1=$\frac{n-1}{2} + 1 = \frac{n+1}{2}$

If n is odd $\left[\frac{n}{2}\right] + 1 = \frac{n}{2} + 1$

From previous article,

$u(x,t;s) \in C^2\left(R^n \times (\delta, \infty)\right) \quad for \quad each\ \delta \geq 0$

so $u \in C^2(R^n \times [0,\infty))$

(ii)

$$u(x,t) : = \int_0^t u\,(x,t;s)\,ds$$

Differentiating w.r.t t

$$u_t(x,t) : = \int_0^t u_t\,(x,t;s)\,ds + u(x,t;t)$$

$$= \int_0^t u_t\,(x,t;s)\,ds \qquad\qquad (by\ 4)$$

Again differentiating w.r.t. t

$$u_{tt}(x,t) := \int_0^t u_{tt}\,(x,t;s)ds + u_t(x,t;t)$$

$$\int_0^t u_{tt}\,(x,t;s)ds + u_t(x,t;t) \qquad\qquad (by\ 4) \qquad\qquad (5)$$

$$\Delta u(x,t) \coloneqq \int_0^t \Delta u\,(x,t;s)ds$$

$$= \int_0^t u_{tt}\,(x,t;s)ds \qquad\qquad (by\ 3) \qquad\qquad (6)$$

$$u_{tt}(x,t) - \Delta u(x,t) = \int_0^t [u_{tt}\,(x,t;s) - \Delta u(x,t;s)]ds + f(x,t)$$

$$= f(x,t)$$

(iii) Also $u(x,0) = 0$

$u_t(x,0) = 0$

The solution of non-homogeneous wave equation is given by equation (3)

Exercise:

1. Find the solution of

$u_{tt} - u_{xx} = 0 \quad in \quad R^n \times (0, \infty)$

$u = g \quad , \quad u_t = h \quad on\, R \times \{t = 0\}$

$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t) + \frac{1}{2}\int_{n-t}^{n+t} h\,(y)dy\right]$

Hence,

$$u(x,t;s) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f\,(y,s)dy$$

(Replacing t by t-s)

Hence

$$u(x,t) = \frac{1}{2}\int_{0}^{t}\int_{x-t+s}^{x+t-s} f\,(y,s)dy\;\,ds$$

Replacing t-s by s , We find

$$u(x,t) = \int_{0}^{t}\int_{x-s}^{x+s} f\,(y,t-s)dy\;\,ds$$

Is the required solution.

Check your progress

2. Find the solution of wave equation for n>3 and derive some identities.

13.6 ENERGY METHOD

Uniqueness of solution

Let $\boldsymbol{U} \subset \boldsymbol{R^n}$ be a bounded, open set with a smooth boundary ∂U and

$$U_T = U(0,T]$$

$$\Gamma_T = \overline{U}_T - U_T \;\; Where \;\; T > 0$$

There exists at most one function $u \in C^2(\overline{U}_T)$ of the initial value problem.

$$u_{tt} - \Delta u = f \qquad in \;\; U_T \qquad (1)$$

$$u = g \ on \ \Gamma_T; \mathrm{U} = \mathrm{h} \text{ on } \mathrm{U} \times \{\mathrm{t} = 0\} \qquad (2)$$

Proof. Let $\bar{u}$ be another solution of equation (1). We take

$$w(x,t) = u - \bar{u},$$

So

$$w_{tt} - \Delta w = 0 \ in \ U_T$$

$$w = 0 \ on \ \Gamma_t; \quad w_t = 0 \ on \ U \times \{t = 0\}$$

Define

$$e(t) := \frac{1}{2}\int_U [{w_t}^2 + |Dw|^2]dx \qquad 0 \le t \le T$$

Differentiating w.r.t. t

$$\dot{e}(t) = \int_U [w_t w_{tt} + Dw\, Dw_t]dx$$

(Integrating the 2nd integral by parts)

$$= \int_U [w_t w_{tt} + D^2 w\, w_t]dx + \int_{\partial U} w_t Dw\, \hat{v}\, dS$$

$$= \int_U w_t\, [w_{tt} - \Delta w]dx + 0 \qquad (using\ 4)$$

$$=0 \qquad \text{(by 3)}$$

So *e(t)* = constant for all *t.*

$$But \ \ e(0) = \frac{1}{2}\int_U [{w_t}^2(x,0) + |Dw(x,0)|^2]dx$$

=0.

So *e(t) is zero for all t.*

$$i.e. Dw \equiv w_t = 0 \ within \ \mathrm{U_T}$$

$$Since \ w = 0 \ on \ U \times \{t = 0\}$$

$$w = u - \bar{u} = 0 \ in \ \mathrm{U_T}$$

$$u = \bar{u} \ in \ \mathrm{U_T}$$

Def. *Let* $u \in \mathrm{C}^2$ be a solution of

$$u_{tt} - \Delta u = 0 \; in \; R^n \times (0, \infty)$$

Fix $x_0 \in R^n, \quad t_0 > 0$

Consider the set

$C = \{(x,t) |\, 0 \le t \le t_0 \,; \quad |x - x_0| \le t_0 - t\}$

Which defines a cone.

Theorem. If

$u \equiv u_t \equiv 0 \; on \; B(x_0, t_0) \times \{t = 0\} then \; u = 0 \; within \; cone \; C$

Proof. We define

$$e(t) = \frac{1}{2} \int_{B(x_0, t_0 - 1)} [{u_t}^2(x,t) + |Du(x,t)|^2] dx \qquad 0 \le t \le t_0$$

$Differentiatinf \; w.r.t.t.$

$$e(t) = \frac{1}{2} \int_{B(x_0, t_0 - 1)} [u_t u_{tt}(x,t) + (Du \; Du_t)] dx$$

$$-\frac{1}{2} \int_{B(x_0, t_0 - 1)} [{u_t}^2 + |Du|^2] ds \qquad \text{(By cor of coarea formula)}$$

Integrating by parts (2nd term of 1st integral)

$$= \int_{B(x_0, t_0 - 1)} [u_t u_{tt} - u_t \, \Delta u)] dx$$

$$+ \int_{\partial B(x_0, t_0 - 1)} \frac{\partial u}{\partial v} u_t \; ds - \frac{1}{2} \int_{\partial B(x_0, t_0 - 1)} [{u_t}^2 + |Du|^2] dS$$

$$= 0 + \int_{\partial B(x_0, t_0 - 1)} \left[u_t \frac{\partial u}{\partial v} - \frac{1}{2} {u_t}^2 - \frac{1}{2} |Du|^2 \right] dx$$

$$\le \int_{\partial B(x_0, t_0 - 1)} \left[{u_t}^2 + |Du|^2 - \frac{1}{2} {u_t}^2 - \frac{1}{2} |Du|^2 \right] dx$$

(by Cauding Schwartz Inequality)

≤ 0

So *e(t) is a* decreasing function of t

$e(t) \leq e(0)$

$But\ e(0) = 0$

$e(t) \leq 0$

hence *e(t)*=0 $(\because e(t) is\ a\ sum\ of\ square\ quantities)$

$u_t = Du = 0\ within\ C$

$\Longrightarrow u$ is constant within C

Hence u=0 within c $(\because u = 0\ for\ t = 0)$

13.7 LET US SUM UP

In this unit we have discussed Solution of 1-D wave equation, Kirchoff's formula, Solution of 2-D wave equation, Solution of wave equation for $n \geq 3$, Solution for odd n, Solution for even n, Solution of Non-homogeneous wave equation and energy methods. The homogeneous wave equation. D-Alembert's formula, Poisson's Formula, Solution for odd n$(n \geq 3)$, solution for even n, If $u \equiv u_t \equiv 0\ on\ B(x_0, t_0) \times \{t = 0\} then\ u = 0\ within\ cone$ C.

13.8 KEY WORDS

1. The general solution of 1-D wave equation

$(u_t + u_x)(u_t - u_x) = 0$

2. Euler Poisson Darboux equation is $u_{tt} - u_{rr} - \frac{n-1}{r} U_r = 0\ \ in\ R_+ \times (0, \infty)$

3. Poisson's formula is $u(x,t) = \frac{1}{2} \oint_{B(x,t)} \frac{t\,g(y)+t^2 h(y)+Dy(y).(y-x)}{\sqrt{t^2-|y-x|^2}} dy$

4. Uniqueness of solution

   Let $\boldsymbol{U} \subset \boldsymbol{R^n}$ be a bounded, open set with a smooth boundary ∂U and $U_T = U(0, T]$

$$\Gamma_T = \overline{U}_T - U_T\ \ Where\ \ T > 0$$

13.9 QUESTION FOR REVIEW

1. Find the solution of

Notes

$\boldsymbol{u_{tt} - \Delta u = f(x,t)\ \ in\ R^3} \times (\mathbf{0}, \infty)$

$u = 0 \quad , \quad u_t = 0 \quad on\ R^3 \times \{t = 0\}$

Ans. $u(x,t) = \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\, dy$

2. Discuss about solution of 1-D wave equation
3. Discuss about solution of 2-D wave equation
4. Discuss about non homogeneous wav equation and energy methods.

13.10 SUGGESTED READINGS AND REFERENCES

1. S. L. Ross, Differential Equations, 3rd Edn., Wiley India, 1984.
2. DiBenedetto, Partial Differential Equations, Birkhaüser, 1995.
3. L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.
4. I.N. Sneddon Elements of Partial Differential Equations McGrawHill 1986.
5. R. Churchil & J. Brown, Fourier Series & Boundary Value Problems.
6. R.C. McOwen , Partial Differential Equations (Pearson Edu.) 2003.
7. Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum, Outline Series, McGraw hill Series.
8. Partial Differential Equations, -Walter A.Strauss
9. Partial Differential Equations,-John K.Hunter
10. Partial Differential Equations,*Erich Mieremann*
11. Partial Differential Equations,-*Victor Ivrii*

13.11 ANSWERS FOR CHECK YOUR PROGRESS

1. See section 13.2
2. See section 13.3
3. See section 13.4
4. See section 13.5

UNIT-14 SEPARATION OF VARIABLES

STRUTURE

14.0 OBJECTIVE

In this unit we will learn understand about Separation of variable, Similarity solutions, Connecting non-linear partial differential equations to linear partial differential equations, Cole-Hopf transformation, Fourier Transforms, Laplace Transforms.

Notes

14.1 INTRODUCTION

There are several other techniques to solve the linear and non- linear partial differential equations. e.g. Separation of variables, Similarity solutions, Connecting non-linear partial differential equations to linear partial differential equations, Transform methods. Here we will discuss them.

14.2 SEPARATION OF VARIABLES

In this method, we assume a solution given by sum or product of undetermined functions and form ordinary differential equations, which are solved.

This technique is well understood by examples.

Exp. Consider the boundary value problem in heat equation

$$u_t - \Delta u = O \ in \ U \times (O, \infty) \quad (1)$$

$$u = O \quad on \quad \partial U \times [0, \infty) \quad (2)$$

$$u = g \ on \ U \times \{t = O\}$$

Where $g : U \to R \quad is \ g$iven.

Sol. Let the solution of equ. (1) be given by

$$u(x,t) = v(t)w(x) \qquad x \in U, t \geq 0 \quad (3)$$

From (1) and (3)

$$vw(x) - \Delta \mathrm{wv(t)} = 0$$

Dividing by $w(x)v(t)$ $\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)}$ (4)

L.H.S. of equ.(4) is a function of t only and R.H.S. is function of x only.

Equ. (4) is true if each side is equal to some constant, say, μ

$$\frac{v'(t)}{v(t)} = \mu = \frac{\Delta w(x)}{w(x)}$$

$$\Longrightarrow v'(t) - \mu v(t) = 0$$

(5)

And

$$\Delta w(x) - \mu w(x) = 0$$

(6)

Considering equ.(5) and integrating

$$v = C\, e^{\mu t}$$

(7)

where C is a constant.

Taking equ. (6), comparing with the

$$\left.\begin{array}{ll} -\Delta \mathrm{w} = \lambda \mathrm{w} & \text{in } U \\ w = O & on\ \partial U \end{array}\right\}$$

(8)

then λ is eigen value and w $(\neq O)$ is the corresponding eigen function. so $\mu = -\lambda$ is eigen value of equ. (6) and wis corresponding eigen function.

Hence solution of problem defined by equ. *(1) and (2) is*

$$\mu\,(x,t) - Ce^{-\lambda t}w(x)$$

(9)

where C is a constant to be determined from the initial condition at t = 0, which gives

$$g = Cw$$

so $\quad u = \mathrm{C}e^{-\lambda t}w$

where $\quad g = Cw$

is required solution.

Particular case:

(a) If $\lambda_{1,}\, \lambda_{2,} \dots, \lambda_m$ are eigen values of problem (8) and $w_1, w_2, \dots, w_m$ are the corresponding eigen functions and $c_{1,}\, c_{2,} \dots, c_m$ are constants then solution of equ. (1)

$$\mathrm{u}\,(x,t) = \textstyle\sum_{i-i}^{m} c_i e^{-\lambda_i t} w_i(x)$$

$$provided \sum_{i=1}^{m} d_i\, w_i = g.$$

(b) *Let $\lambda_{1,} \lambda_{2,} \ldots,$ be a countable set of eigen values with corresponding eigen function $w_1, w_2, \ldots$ so that*

u $= \sum_{i-1}^{\infty} c_i e^{-\lambda_i t} w_i(x)$ Provided that $\sum_{i-1}^{\infty} c_i w_i(x) = g \; iin \; U.$

Exp. Find the solution of the non-linear porous medium equation

$$u_t - \Delta(u^\gamma) = 0 \; in \; R^n \times (0, \infty) \quad (1)$$

Where $u \geq 0 \; and \; \gamma > 1 \; is \; a \; constant$

Sol. We seek a solution of equ. (1) of the type

$$u(x,t) = v(t)w(x) \quad (2)$$

From (1) and (2)

$$w(x)v'(t) - (\Delta w^\gamma)v^\gamma = 0$$

Dividing by wv^γ

$$\frac{v'(t)}{v^\gamma} = \frac{\Delta w^\gamma}{w}$$

(3)

L.H.S. is a function of t only and R.H.S. is a function of x only. Equ.3 is true if each side is equal to some constant say μ

$$\frac{v'(t)}{v^\gamma} = \mu$$

$$\frac{v^{\gamma+1}}{-\gamma+1} = \mu t + \lambda,$$

Where λ is a constant.

$$v^{1-r} = (1-\gamma)\mu t + \lambda,$$

$$v = [(1-\gamma)\mu t + \lambda]^{\frac{1}{1-\gamma}}$$

(4)

$$\Delta w^\gamma = \mu w$$

(5)

Suppose $w = |x|^a$ is solution of equ. (5) where α is a constant to be determined.

$$\Delta w^\gamma = \Delta |x|^{\alpha\gamma}$$

$$= |x|^{\alpha y - 2} \; \alpha\gamma[n + \alpha\gamma - 2]$$

Using in equ. (5)

$$|x|^{\alpha y - 2} \; \alpha\gamma[n + \alpha\gamma - 2] = \mu |x|^\alpha$$

(6)

In order to hold equ. (6) in $R^n, we \; must \; have$

$$\alpha = \alpha\gamma - 2 \qquad \Rightarrow \ \alpha = \frac{2}{\gamma - 1} \ and$$

$$\mu = \ \alpha\gamma(n + \alpha\gamma - 2) > 0$$

(7)

So solution of equ. (1) is

$$u = [(1-\gamma)\mu t + \ \lambda]^{\frac{1}{1-\gamma}} |x|^{\alpha}$$

(8)

Where α, μ are given by equ. (7)

Remark. In equ. (8) u is singular when

$$(1-\gamma)\mu t + \lambda \ = 0$$

$t = \dfrac{\lambda}{(\gamma - 1)\mu} = t^*$ (say) , t^* is called the critical time.

Check your progress

1. Explain about Separation of variables.

----

2. Find the solution of the non-linear porous medium equation

$u_t - \Delta(u^{\gamma}) = \ 0 \ in \ R^n \ \times (0, \infty)$

Where $u \geq 0 \ and \ \gamma > 1 \ is \ a \ constant$

14.3 SIMILARITY SOLUTION

Certain symmetries of partial differential equation help to convert them in ordinary differential equations.

Def. Plane Travelling Wave :

A Solution $u(x,t)$ of the partial differential equations of two variables $x, t \ \in R$ of the form

$$u(x,t) = \ v(x - \sigma t) \qquad x \ \in \ R, \quad t \in R$$

Represents a travelling wave with speed σ and velocity profile v.

Generalization. A Solution $u(x,t)$ of a partial differential equations in $n+1$ variables $x = (x_1, x_2, \dots, x_n) \in R^n, t \in R$ having the form

$$u(x,t) = v(y.x - \sigma t)$$

Is called a plane wave with wave front normal to $y \in R^n$.

Exponential Solution : The exponential solution of partial differential equations is

$u(x,t) = e^{i(yx;+wt)}$

Where $w \in R, y = (y_1, y_{2,\ldots,}y_n) \in R^n, w$ being frequency and$\{y_i\}_{i=1}^n$ the wave number.

Exp. The heat equation

$u_t - \Delta u = 0$

has the exponential solution.

$u = e^{i(yx+i|y|^2 t)}$

$= e^{-|y|^2 t}[e^{iyx}]$

$e^{-|y|^2 t} \cos yx$ and $e^{-|y|^2 t} \sin yx$ $are\ solutions\ of\ equ.(1).$

Here the term

$e^{-|y|^2 t}$ corresponds the dissipation of energy.

Exp. The wave equation

$u_{tt} - \Delta u = 0$ has exponential solution

$$u = e^{i(y \cdot x \pm |y|^t)}$$

Since w is real, no dissipation effects occur.

Exp. The dispersive equation

$u_t = u_{xxx} = 0 in R \times (0,\infty)$

has the exponential solution

$u = e^{i(y \cdot x + y^3 t)}$

No dissipation of energy. Also the velocity of propagation depends on frequency. Hence dispersion takes place.

Exp. Barenblaltt's Solution

Consider the porous medium equation

$u_t - \Delta u^\gamma = o in R^n \times (0,\infty)$ (1)

Where $u \geq 0$ and $\gamma > 1$ is a constant.

Sol. We seek a solution of equ.(1) of the form

$$u(x,t)=\frac{1}{t^{\propto}}v\left(\frac{x}{t^{\beta}}\right) \qquad x\in R^n,\quad t>0$$

Where α,β are un knowns.

$$u(x,t)=\frac{1}{t^{\alpha}}v(y)$$

Where $y=x/t^{\beta}$

From equation (1) and (2)

$$\alpha v(y)+\beta y.Dv(y)+\frac{1}{t^{\alpha\gamma+2\beta-\alpha-1}}v^{\gamma=}0$$

To Convert equ.(3) into an equation independent of t, we must have

$\alpha\gamma+2\beta-\alpha-1=0$ or

$$\alpha=\frac{1-2\beta}{\gamma-1}$$

Hence equ. (3) gives

$$\alpha v+\beta y.Dv+\Delta v^{\gamma}=0$$

We seek a radial solution of equ.(5)

Let it be

$v(y)=w(r)$ where $r=|y|$

From equ.(5) and (6)

$$\alpha w+\beta w'(r)r+(w^{\gamma})''+(n-1)(w^{\gamma})'r^{n-2}=0$$

Where dash denotes derivatives w.r.t. r.

To make it exact differential, Multiplying by γ^{n-2} and taking

$\alpha=n\beta$ 1

$$\beta(r^n w)'+\left[r^{n-1}(w^{\gamma})'\right]'=0$$

Integrating and assuming that as $r\to 0$ $w,w'\to 0$

$$\beta r^n w+r^{n-1}(w^{\gamma})'=0$$

$$(w^{\gamma})'=-\beta rw$$

$$\gamma w^{\gamma-1}=-\beta rw$$

Or $w^{\gamma-2}=\dfrac{-\beta}{\gamma}r$

Again integrating

$$w^{\gamma-1} = \frac{-\beta r^2}{2\gamma(\gamma-1)} + b$$

Where b is a constant.

$$\Rightarrow w(r) = \left[b - \frac{(\gamma-1)\beta}{2\gamma} r^2 \right]^{\frac{1}{\gamma-1}}$$

Hence

$$v(y) = \left[b - \frac{(\gamma-1)\beta |x|^2}{2\gamma t^{2\beta}} \right]^{\frac{1}{\gamma-1}}$$

Where

$$\alpha = n\beta = \frac{1-2\beta}{\gamma-1}$$

I.E. $\beta = \dfrac{1}{2-n+n\gamma}$

$$\alpha = \frac{n}{2-n+n\gamma}$$

Check your progress

Explain about Similarity solutions.

14.4 CONNECTING NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS TO LINEAR PARTIAL DIFFERENTIAL EQUATIONS

14.4.1 Cole-Hopf transformation

Consider the initial value problem for a quasi-linear parabolic equation

$$u_t - a\Delta u + b|Du|^2 = 0 \quad \text{in } R^n \times (0,\infty) \qquad (1)$$

$$u = g \quad \text{on } R^n \times \{t=0\} \qquad (2)$$

Where $a > 0; a, b$ are constants.

Sol. Let $w = \varphi(u)$ (3)

Where u is a smooth solution of equ.(1) and $\varphi : R \to R$ is a smooth function. We seek φ such that w solves the linear equation.

From (1) and (3)

$$Dw = \varphi'(u)Du$$

$$w_t = \varphi'(u)\left[a\Delta u - b|Du|^2\right]$$

$$= a\left[\Delta w - \varphi''(u)|Du|^2\right] - b\varphi'(u)|Du|^2$$

$$Hence, w_t - a\Delta w = -\left[a\varphi''(u) + b\varphi'(u)\right]|Du|^2$$

We choose φ such that

$$a\varphi''(u) + b\varphi'(u) = 0 \qquad (4)$$

So we have

$$w_t - a\Delta w = 0 \qquad (5)$$

To find the solution of equ. (4)

Auxiliary equation is $am^2 + bm = 0$

Roots with $m = 0, -b/a$

Hence

$\varphi(u) = e^{-(b/a)u} + C,$ where C is constant.

Neglecting the constant

$$w(x,t) = e^{-(b/a)u} \qquad (*)$$

$$w(x,0) = e^{-(b/a)g} \qquad (6)$$

Combing (5) and (6)

$w_t - a\Delta w = 0$ in $R^n \times (0,\infty)$

$w = e^{-\frac{b}{a}g}$ on $R^n \times \{t = 0\}$

Which is heat equation having the solution

$$w(x,t) = \frac{1}{(4\pi at)^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4at}} e^{\frac{-b}{a}g} dy \qquad x \in R^n$$

Or $\quad u(x,t) = -\frac{a}{b}\log w$

$$u(x,t) = -\frac{a}{b}\log\left[\frac{1}{(4\pi at)^{n/2}}\int_{R^n} e^{\frac{-|x-y|^2}{4at}} e^{\frac{-b}{a}g} dy\right] \qquad x \in R^n, t > 0$$

This is the required Solution. Equation (*) is known as Cole Hopf transformation.

Exp.Find the solution of Burger's Equation with Viscosity

$$u_t - a\ u_{xx} + u\ \ u_x = 0 \text{ in } R\times(0,\infty)$$

$$u = g \quad \text{on } R\times\{t=0\} \qquad (1)$$

Sol. Let us take

$$w(x,t) := \int_{-\infty}^{x} u(y.t)dy$$

$$h(x) := \int_{-\infty}^{x} g(y)dy \qquad (2)$$

So that $w_x = u, w_x(x,0) = u(x,0) = g(x) = h'(x)$

Fromo (1) and (3)

$$w_{xt} - a\ \ w_{xxx} + w_x\ \ w_{xx} = 0 \text{ in } R\times(0,\infty)$$

$$\frac{\partial}{\partial x}\left[w_t - a\ w_{xx} + \frac{1}{2}w^2{}_x\right] = 0 \quad \text{in } \mathrm{R}\times(0,\infty)$$

Thus, problem is converted to

$$w_t - a\ w_{xx} + \frac{1}{2}w^2{}_x = 0 \text{ in } \mathrm{R}\times(0,\infty)$$

$$w(x,0) = h(x) \quad \text{on } \mathrm{R}\times\{t=0\} \qquad (4)$$

The equation (4) is a quasi –linear parabolic equation (previous example) with b=$\frac{1}{2}$. So solution of equ.(4) is

$$w(x,t) = -2a\log\left[\frac{1}{(4\pi at)^{n/2}}\int_{R^n} e^{\frac{-|x-y|^2}{4at}} e^{\frac{-b}{a}g} dy\right]$$

Differentiating w.r.t. x

$$u = w_x = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{\frac{-|x-y|^2}{4at}\frac{h(y)}{2a}dy}}{\int_{-\infty}^{\infty} e^{\frac{-|x-y|^2}{4at}\frac{h(y)}{2a}dy}}$$

This is required solution

14.4.2. Potential Function

By use of potential function, non-linear partial differential equation can be converted to linear partial differential equations.

Exp. Consider the Euler's equation for inviscid, incompressible flow

$$u_t + u.Du = -Dp + f \quad \text{in } R^3 \times (0,\infty)$$

$$div\ u = 0 \text{ in } R^3 \times (0,\infty)$$

$$u = g \text{ on } R^3 \times (t=0) \qquad (1)$$

Where f and g are prescribed functions, u and p are unknowns.

Sol. Let the external body force be derived from potential function *h,*

Such that

$$f = Dh$$

Let the Velocity u be derived from the potential v s.t.

$$u = Dv$$

From equ.(1) and (3)

$$div\ u = \Delta v = 0$$

So from equ. (4) we can find *v and thus* <u>*u.*</u>

From (1) and (3)

$$Dv_t + Dv\ D(Dv) = \text{-}Dp + Dh$$

Or

$$D\left[v_t + \frac{1}{2}|Dv|^2 + p - h\right] = 0$$

Integrating

$$v_t + \frac{1}{2}|Dv|^2 + p = h$$

Which is Bernoulli's equation to get *p.*

14.5 TRANSFORM METHODS

14.5.1 Fourier Transforms

We now discus the transform methods to solve linear and non-linear partial differentiation equations. First we define Fourier transform over L^1 and L^2 spaces, respectively.

Def. Let $u \in L^1(R^n)$, we define the Fourier transform of $u(x)$, denoted by $u(y)$ *as*

$$u(y) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-i\,x.y}\ u(x)dx \qquad y \in R^n$$

and its inverse Fourier transform

$$u(y) : \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{ix.y}\ u(x)dx \qquad y \in R^n$$

Since $\left|e^{\pm i\,xy}\right| = 1$ and $u \in L^1(R^n)$

So integral converges for each y.

Plancherel's theorem

Assume that $u \in L^1(R^n) \cap L^2(R^n)$ then

$\hat{u},\ \breve{u} \in L^2(R^n)$ and

$$\|\hat{u}\|_{L^2(R^n)} = \|\breve{u}\|_{L^2(R^n)} = \|u\|_{L^2(R^n)} \qquad (1)$$

Proof. To Prove (1). We prove three results

(i) $\int_{R^n} v(y)\ \hat{w}(y)\ dy = \int_{R^n} \hat{v}(x)\ w(x)\ dx$

L.H.S.= $\frac{1}{(2\pi)^{n/2}} \int_{R^n} v(y) \int_{R^n} [w(x)e^{-ix.y}\ dx]dy$

$$\frac{1}{(2\pi)^{n/2}} \int_{R^n} w(x) \int_{R^n} (y)e^{-ixy}\ dx\ dy$$

$$\int_{R^n} w\ (x)\ \hat{v}(x)\ dx$$

Hence the result.

(ii) If $u,\ v \in L^1(R^n) \cap L^2(R^n)$

Then $(u * v) = (2\pi)^{n/2}\hat{u}\hat{v}$

Where * denotes the convolution operator.

By def.

$$u*v=\int_{R^n} u(z)v(x-z)dz$$

$$(u*v)^{\wedge}=\frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{-i\,x\,y}\left\{\int_{R^n} u(x)v(x-z)\right\}dx$$

$$=\frac{1}{(2\pi)^{n/2}}\int_{R^n} v(x-z)e^{-iy(x-z)}dx\int_{R^n} e^{-izy}u(z)dz$$

$$=\hat{v}(y)\int_{R^n} e^{-izy}u(z)dz$$

$$=\hat{v}(y)\left[u(y)(2\pi)^{n/2}\right]$$

$$=(2\pi)^{n/2}\hat{u}(y)\hat{v}(y)$$

(iii) Consider

$$\int_{R^n} e^{-ixy-t}\,|x|^2\,dx=\prod_{i=1}^{n}\int_R e^{-x_iy_i-tx_i^2}dx$$

$$\text{But}\int_R e^{-ix_iy_i-tx_i^2}dx_i=\int_{-\infty}^{\infty} e^{-t\left[-tx_i^2+\frac{ix_iy_i}{t}\right]}dx$$

$$=\int_{-\infty}^{\infty} e^{-t\left(x_i+\frac{iy_i}{2t}\right)^2+t\left(\frac{iy_i}{2t}\right)^2 dx_i}$$

$$=\frac{e^{-y_i^2/4t}}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-z^2}dz \quad \text{where } z=\sqrt{t}\left(x_i+\frac{iy_i}{2t}\right)$$

$$=\frac{e^{-y_i^2/4t}}{\sqrt{t}}\sqrt{\pi}$$

Hence $\displaystyle\int_{R^n} e^{-ixy-t|x|^2}dx=\left(\frac{\pi}{t}\right)^{n/2}e^{-|y|^2/4t}$

Proof of theorem

Choosing a function for $\in>0$

$$v_\in(x)=e^{-\in|x|^2}$$

$$\hat{v}_\in(y)=\frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{-ixy}v_\in(x)dx \text{ (using result (iii), putting } t=\in)$$

$$=\frac{1}{(2\in)^{n/2}}e^{-|y|^2/4\in} \qquad (2)$$

Hence

$$\int_{R^n} \hat{w}(y)e^{-\in|y|^2}dy = \frac{1}{(2\in)^{n/2}}\int_{R^n} w(x)e^{-|x|^2/4\in}dx \quad \text{(using result (i))}$$

(3)

Taking limit as $\in \to 0$

$$\int_{R^n} \hat{w}(y)\ dy = \underset{\in\to 0}{lt}\frac{1}{(2\in)^{n/2}}\int_{R^n} w(x)e^{-|x|^2/4\in}dx$$

$$= (2\pi)^{n/2}w(0) \quad \text{where } \frac{x_i^2}{4\in} = Z_i^2 \qquad (4)$$

Suppose $u \in L^1(R^n) \cap L^2(R^n)$

and set $v(x) := \bar{u}(-x)$, $\bar{u}$ is the conjugate if u .

$$w(x) := u * v$$

$$= \int_{R^n} u(z)v(x-z)dz$$

$$\hat{w} = (2\pi)^{n/2}\hat{u},\hat{v} \qquad \text{(by result II)}$$

But $\hat{v} = \dfrac{1}{(2\pi)^{n/2}}\displaystyle\int_{R^n} e^{-ixy}\bar{u}(-x)dx$

$$= \frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{-i\,x\,y}\bar{u}(-x)dx$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{R^n} \overline{e^{-i\,x\,y}u(-x)}\ dx$$

$$= \overline{\hat{u}(y)}$$

$$\therefore \hat{w} = (2\pi)^{n/2}|\hat{u}|^2$$

From (4) and (5)

$$(2\pi)^{n/2}\int_{R^n}|\hat{u}|^2dy = (2\pi)^{n/2}w(0)$$

Or $\displaystyle\int_{R^n}|\hat{u}|^2dy = \int_{R^n} u(z)\bar{u}(z)dz$

$$= \int_{R^n}|\hat{u}|^2dz \qquad \text{(by def.)}$$

$$\|\hat{u}\|_{L^2(R^n)} = \|u\|_{L^2(R^n)}$$

Similarly $\quad \|\breve{u}\|_{L^2(R^n)} = \|u\|_{L^2(R^n)}$

(the result can be obtained by previously argument changing *i* to *–i)*

Hence $\|u\|_{L^2(R^n)} = \|\hat{u}\|_{L^2(R^n)} = \|\breve{u}\|_{L^2(R^n)}$

Note

Since $u \in L^2(R^n)$ choose a sequence $\{\mathrm{u_k}\}_{k=1}^{\infty} \subset L^1(R^n) \cap L^2(R^n)$ with

$u_k \to u$ in $L^2(R^n)$.

By (1)

$$\|u_k - u_j\|_{L^2(R^n)} = \|\hat{u}_k - \hat{u}_j\|_{L^2(R^n)} = \|u_k - u_j\|_{L^2(R^n)}$$

$\{\hat{u}_k\}_{k=1}^{\infty}$ is a cauchy sequence in $L^2(R^n)$ which converges to $\hat{u}$.

So $\hat{u}_k \to \hat{u}$ in $\mathrm{L^2(R^n)}$

Def. Fourier Transform of u over $L^2\left(R^n\right)$.

Let $u \in L^2(R^n)$ then

$$\|\hat{u}\|_{L^2(R^n)} = \|\breve{u}\|_{L^2(R^n)} = \|u\|_{L^2(R^n)}$$

So $\hat{u}, \breve{u} \in L^2(R^n)$ (by above theorem)

Hence $\hat{u}, \breve{v}$ are well defined over $L^2\left(R^n\right)$.

Properties of Fourier transform:

Assume $u, v \in L^2\left(R^n\right)$

(i) $\displaystyle\int_{R^n} u\ \bar{v}\ dx = \int_{R^n} \hat{u}\ \bar{\hat{v}}\ dy$

(ii) $D^{\alpha}u = (iy)^{\alpha}\hat{u}$

for each multiindex α s.t. $D^{\alpha}u \in L^2(R^n)$

Proof. Let $u, v \in L^2(R^n)$ and $\alpha \in C^n$

Then

$\|u + \alpha v\|^2 = \|\hat{u} + \alpha\hat{v}\|^2$ (Usiing plancherel's theorem)

i.e. $\displaystyle\int_{R^n} (u + \alpha v)(\overline{u} + \overline{\alpha v})dx = \int_{R^n} (\hat{u} + \alpha\hat{v})((\bar{\hat{u}} + \bar{\alpha}\bar{\hat{v}}))dy$

$$\Rightarrow \int_{R^n} \left[|u|^2 + |\alpha v|^2 + u(\alpha v) = (\alpha v) + u(\alpha\, v)\right]dx$$

$$= \int_{R^n} \left[|\hat{u}|^2 + |\alpha\hat{v}|^2 + \bar{\hat{u}}(\alpha\hat{v}) + \hat{u}(\bar{\alpha}\,\bar{\hat{v}})\right]dy$$

Or $\displaystyle\int_{R^4} \left[\overline{u}(\alpha\, v) + u(\overline{\alpha v})\right]dx = \int_{R^n} (\alpha\bar{\hat{u}}\hat{v} + \bar{\alpha}\hat{u}\bar{\hat{v}})dy$ (1)

Notes

Taking $\alpha = 1$ in (1) respectively and subtracting we obtain

$$\int_{R^n} u\bar{v}\ dx = \int_{R^n} (\hat{u}\bar{\hat{v}})\ dy$$

(ii) If u is smooth and has compact support

$$D^{\alpha}u = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-ix\,y} D^{\alpha}u\ dx$$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{R^n} D^{\alpha} e^{-ixy} u(x)\ dx$$

$$= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{R^n} e^{-ixy} (-1)^{\alpha} (iy)^{\alpha} u(x)\ dx$$

$$= (iy)^{\alpha}\hat{u}(y)$$

Exp. Solve the partial differential equation

$$-\Delta u + u = f \text{ in } R^n \qquad \textbf{(1)}$$

Where $f \in C^2(R^n)$

Sol. Taking Fourier transform of equation (1)

$$-(iy)^2\hat{u} + \hat{u} = \hat{f}, \qquad y \in R^n$$

$$\hat{u} = \frac{\hat{f}}{1+y^2}$$

Taking inverse Fourier transform of (2)

$$u = \left(\frac{\hat{f}}{1+y^2}\right)^{v}$$

$u = f * B$ where

$$B = \frac{1}{(1+y^2)^{v}}$$

To find B, we know that

$$\frac{1}{a} = \int_0^{\infty} e^{-ta} dt$$

So $$\frac{1}{1+|y|^2} = \int_0^{\infty} e^{-t\left(1+|y|^2\right)} dt$$

$$\Rightarrow \left(\frac{1}{1+|y|^2}\right)^{v} = \frac{1}{(2\pi)^{n/2}} \int_{R^n}\int_0^{\infty} e^{-t\left(1+|y|^2\right)} e^{ixy} dt\ dy$$

$$= \frac{1}{2^{n/2}} \int_0^\infty e^{-t\left(\frac{\pi}{t}\right)^{n/2}} e - |x|^{2/4t} \, dt$$

$$= \frac{1}{(2\pi)^{n/2}} \int_0^\infty \frac{e^{-t} \frac{-|x|^2}{4t}}{t^{n/2}} \, dt \qquad x \in R^n \qquad (3)$$

So,

$$u(x, y) == \frac{1}{(4\pi)^{n/2}} \int_0^\infty \int_{R^n} \frac{f(y) e^{-t} - \frac{|x|^2}{4t}}{t^{n/2}} \, dy \, dt$$

Here, B given in equation (3), is called the Bessel's potential.

Exp. Find the solution of initial value problem o f heat equation.

$$u_t - \Delta u = 0 \ \text{ in } R^n \times (0, \infty)$$

$$u = g \ \text{ on } R^n \times \{t = 0\}$$

Sol. Taking Fourier transform of equation (1) and (2) w.r.t the spatial variable *x*.

$$\hat{u}_t - (iy)^2 \hat{u} = 0 \quad \textit{for } t > 0 \qquad (3)$$

$$\hat{u} = \hat{g} \qquad \text{for } t = 0 \qquad \textit{(4)}$$

Or

$$\frac{\hat{u}_t}{\hat{u}} = -y^2$$

Integrating

$\hat{u} = Ce^{-y^2 t}$, where C is a constant.

Since $C = \hat{g}$ (using (4))

$$\hat{u} = \hat{g} e^{-|y^2| t}$$

Taking inverse Fourier transform

$$u = \frac{g * F}{(2\pi)^{n/2}}$$

Where,

$$F = \left(e^{-t|y|^2}\right)^{\text{v}}$$

$$= \frac{1}{(2t)^{n/2}} e - |x|^{2/4t}$$

Hence solution is

$$u(x,y)=\frac{1}{(4\pi)^{n/2}}\int_{R^n} g(y)e\frac{-|x-y|^2}{4t}dy$$

Exp. Solve the Schrödinger's equation.

$$iu_t+\Delta u=0 \;\text{ in } R^n\times(0,\infty) \qquad (1)$$

$$u=g \quad on\ R^n\times\{t=0\} \qquad (2)$$

Where *u* and *g* are complex valued functions.

Sol. Equ.(1) can be rewritten as

$$\frac{\partial u}{\partial(-it)}+\Delta u=0$$

Which is obtained from heat equation replacing *t by it. Hence we get*

$$u(x,y)=\frac{1}{(4\pi it)^{n/2}}\int_{R^n} e^{\frac{i|x-y|^2}{4t}}g(y)\,dy \quad \mathrm{t}\neq 0 \qquad (3)$$

Which is required solution.

Remark. From equ. (3), we can obtain the fundamental solution of Schrodinger equation.

$$\psi(x,y)=\frac{1}{(4\pi it)^{n/2}}\int_{R^n} e^{\frac{i|x|^2}{4t}} \qquad \mathrm{x}\in R^{n,} \quad \mathrm{t}\neq 0$$

Exp. Find the solution of initial value problem

$$u_{tt}-\Delta u=0 \;\text{ in } R^n\times(0,\infty) \qquad (1)$$

$$\left.\begin{array}{l}u=g\\ u_t=0\end{array}\right\} \qquad \text{on } \mathrm{R}^{\mathrm{n}}\times\{t=0\} \qquad (2)$$

Sol. Taking Fourier transform of equation (1) w.r.t. *x*

$$\hat{u}_{tt}+|y|^2\hat{u}=0 \quad \text{for } t>0 \qquad (3)$$

$$\hat{u}=\hat{g} \quad \hat{u}=0 \quad \text{for } t=0 \qquad (4)$$

We seek an exponential solution of equ.(3). Let

$\hat{u}=\beta e^{t\gamma}$ where $\beta,\gamma\in\ R$ are to be determined.

From (1) and (3)

$$|\gamma|^2+|y|^2=0$$

$$\gamma=\pm i|y|$$

$$\hat{u}=\beta_1 e^{i|y|t}+\beta_2 e^{i|y|t}$$

Using equation (2), we obtain

$$\beta_1 - \beta_2 = 0$$

$$\Rightarrow \beta_1 = \beta_2 = \beta \quad \text{(say)}$$

And $\beta = \hat{g}/2$

Hence,

$$u(x,t) = \frac{\hat{g}}{2}(e^{i|y|t} + e^{-i|y|t})$$

Taking inverse Fourier transform

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{R^4} \frac{\hat{g}}{2}(e^{i|y|t} + e^{-i|y|t})e^{ixy}dt \qquad x \in R^n, \quad t \geq 0$$

Is the required solution.

14.5.2 Laplace Transforms

Laplace transform method is useful for functions defined on R_+ i,e.

$(0,\infty)$ if $u \in L^1(R_+)$ we define the Laplace transform of u

$$L(\bar{u}(s)) := \int_0^\infty e^{-st}u(t)dt \qquad s \geq 0$$

We denoted by $\bar{u}$.

Exp. Solve the heat equation

$$v_t - \Delta v = 0 \ \text{ in } U \times (0,\infty) \tag{1}$$

$$v = f \qquad on\ U \times \{t = 0\} \tag{2}$$

Sol. Taking Laplace transform of (1) w.r.t. t

$$\Delta\bar{v}(x,s) = \int_0^\infty e^{-st}\Delta v(x,t)dt$$

$$= \int_0^\infty e^{-st}v_t(x,t)dt$$

$$= e^{-st}v(x,t)\Big|_0^\infty + \int_0^\infty se^{-st}v(x,t)dt$$

$$= e^{-st}v(x,t)\Big|_0^\infty + s\bar{v}(x,s)$$

$$= -f(x) + s\bar{v}(s)$$

Hence

$$-\Delta\bar{v}(s) + s\bar{v}(s) = f \tag{3}$$

Equation (3) is called Resolvent equation. The solution of resolvent equation is Laplace transform of equation (1).

EXERCISE:

1. Solve the Hamilton Jacobi equation

$$u_t - H(Du) = 0 \text{ in } R^n \times (0,\infty)$$

Where H is the Hamilton function.

2. Find the exponential solution of Schrodinger's equation

$$i\, u_t + \Delta u = 0 \text{ in } R^n$$

3. Solve the telegraph equation.

$$u_{tt} + 2du_t - u_{xx} = 0 \text{ in } R^n \times (0,\infty)$$

$$u = g \quad u_t = h \text{ on } R \times \{t = 0\}$$

For $d > 0$.

4. Prove that

(i) if $u, v \in L^1(R^n) \cap L^2(R^n)$ then $(u * v)^n = (2\pi)^{n/2}\, \hat{u}, \hat{v}$

(ii) $u = (\hat{u})^{\vee}$

14.6 LET US SUM UP

In this unit we have discussed about Separation of variable, Similarity solutions, Connecting non-linear partial differential equations to linear partial differential equations, Cole-Hopf transformation, Fourier Transforms, Laplace Transforms. Sum or product of undetermined functions and form ordinary differential equations. Symmetries of partial differential equation help to convert them in ordinary differential equations. The exponential solution of partial differential equations. Use of potential function, non-linear partial differential equation can be converted to linear partial differential equations.

14.7 KEY WORDS

1. Certain symmetries of partial differential equation help to convert them in ordinary differential equations.
2. The exponential solution of partial differential equations.
3. The heat and wave equation
4. By use of potential function, non-linear partial differential equation can be converted to linear partial differential equations.
5. Properties of Fourier transforms

6. Properties of Lapalce transforms.

14.8 QUESTIONS FOR REVIEW

1. Discuss about Separation of variables.
2. Discuss about connecting non-linear partial differential equations to linear partial differential equations.
3. Discuss about Laplace Transforms

14.9 SUGGESTED READINGS AND REFERENCES

1. S. L. Ross, Differential Equations, 3rd Edn., Wiley India, 1984.
2. DiBenedetto, Partial Differential Equations, Birkhaüser, 1995.
3. L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.
4. I.N. Sneddon Elements of Partial Differential Equations McGrawHill 1986.
5. R. Churchil & J. Brown, Fourier Series & Boundary Value Problems.
6. R.C. McOwen , Partial Differential Equations (Pearson Edu.) 2003.
7. Duchateau and D.W. Zachmann, "Partial Differential Equations," Schaum, Outline Series, McGraw hill Series.
8. Partial Differential Equations, -Walter A.Strauss
9. Partial Differential Equations,-John K.Hunter
10. Partial Differential Equations,*Erich Mieremann*
11. Partial Differential Equations,-*Victor Ivrii*

14.10 ANSWERS TO CHECK YOUR PROGRESS

1. See section 14.2
2. See section 14.2
3. See section 14.3